VAN LAMBALGEN'S THEOREM AND HIGH DEGREES

JOHANNA N.Y. FRANKLIN AND FRANK STEPHAN

Abstract. We show that van Lambalgen's Theorem fails with respect to recursive randomness and Schnorr randomness for some real in every high degree and provide a full characterization of the Turing degrees for which van Lambalgen's Theorem can fail with respect to Kurtz randomness. However, we also show that there is a recursively random real that is not Martin-Löf random for which van Lambalgen's Theorem holds with respect to recursive randomness.

1. Introduction

Martin-Löf randomness is the most frequently studied randomness notion. It can be easily defined in terms of unpredictability, measure theory, or initial-segment complexity [18], and in each of these frameworks, there is a universal test, i.e., a single martingale, test, or oracle machine that can be used to determine whether a real is Martin-Löf random [10, 4, 14].

However, there are other well-known notions of randomness, such as recursive randomness, Schnorr randomness, and Kurtz randomness. As we compare and contrast these notions with Martin-Löf randomness and with each other, it is instructive to investigate the extent to which results that are true of one notion are true of the others. We present here some comparative results concerning van Lambalgen's Theorem, a classic theorem in the study of Martin-Löf randomness.

1.1. **Background.** Our notation is standard and generally follows [16, 17] and [19]. For background on effective randomness, please refer to [4] or [14].

In this paper, we will primarily use the unpredictability approach to randomness. In the most general sense, a real, or an element of the Cantor space $\{0,1\}^\omega$, is considered to be random if no algorithm of the appropriate computational strength can predict its $(n+1)^{st}$ bit given its first n bits. We formalize this notion using martingales.

Definition 1.1. A *martingale* is a function $m : \{0,1\}^{<\omega} \to \mathbb{R}^{\geq 0}$ such that for all $\sigma \in \{0,1\}^{<\omega}$,

$$m(\sigma) = \frac{m(\sigma 0) + m(\sigma 1)}{2}.$$

We say that a martingale m is *r.e.* if the values $m(\sigma)$ are uniformly left-r.e. reals and *recursive* if the values $m(\sigma)$ are uniformly recursive reals.

Definition 1.2. A martingale m *succeeds* on a real A if $\limsup_n m(A{\upharpoonright}n) = \infty$.

In other words, a martingale succeeds on a real if there is no bound on the amount of capital the martingale attains by "betting" on the real.

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Martin-Löf, recursive, and Schnorr randomness can all be defined in terms of martingales as below. Recall that an order function is simply a recursive function that is nondecreasing and unbounded.

Theorem 1.3. [18] *Let A be a real.*

1. *A is Martin-Löf random if no r.e. martingale succeeds on it.*
2. *A is recursively random if no recursive martingale succeeds on it.*
3. *A is Schnorr random if there is no recursive martingale m such that for some order function h, $m(A\restriction n) \geq h(n)$ for infinitely many n.*

It is easy to see that every Martin-Löf random real is recursively random and that every recursively random real is Schnorr random. However, neither of the reverse implications hold [18, 21].

In addition to the properties mentioned earlier, Martin-Löf randomness also satisfies several of our intuitions about randomness; e.g., the halves of any random real should not only be random themselves, but also random with respect to each other. Van Lambalgen proved in [20] that the join of any two reals that are Martin-Löf random with respect to each other is a Martin-Löf random real as well.

Theorem 1.4 (van Lambalgen's Theorem)**.** *Let A_0 and A_1 be reals. Then $A_0 \oplus A_1$ is Martin-Löf random if and only if A_0 is Martin-Löf random and A_1 is Martin-Löf random with respect to A_0.*

Since for any two Martin-Löf reals A_0 and A_1, $A_0 \oplus A_1$ is Martin-Löf random exactly when $A_1 \oplus A_0$ is Martin-Löf random, we can see that A_0 is Martin-Löf random with respect to A_1 if and only if A_1 is Martin-Löf random with respect to A_0.

We say that a real $A = A_0 \oplus A_1$ satisfies van Lambalgen's Theorem with respect to a particular randomness notion $\mathcal{R}$ precisely when A is $\mathcal{R}$-random and A_i is $\mathcal{R}$-random with respect to A_{1-i} for each $i \leq 1$. If there is a real A that is $\mathcal{R}$-random for which this statement does not hold, we say that van Lambalgen's Theorem fails for A with respect to $\mathcal{R}$-randomness.

2. Previous work

As mentioned in [22] and [4], one direction of van Lambalgen's Theorem holds for both recursive and Schnorr randomness.

Theorem 2.1. *If A_0 is recursively (Schnorr) random and A_1 is recursively (Schnorr) random relative to A_0, then $A_0 \oplus A_1$ is recursively (Schnorr) random.*

We note that full proofs of these results are not given in either of these references, though Downey and Hirschfeldt indicate that they follow from the test definitions of recursive and Schnorr randomness [4]. We provide a proof of this theorem for the case of Schnorr randomness that is based on the martingale characterization here. Throughout the proof, we will use the notation $(A \oplus \sigma)\restriction k$, where A is a real and σ is a finite binary string. This should be taken to mean the finite binary string resulting from the join of $A\restriction|\sigma|$ and σ for $k \leq 2|\sigma|$; we will never use it for any k larger than this.

Proof. Suppose that A is Schnorr random and that B is Schnorr random relative to A. We will assume that $A \oplus B$ is not Schnorr random and derive a contradiction.

If $A \oplus B$ is not Schnorr random, it follows from Proposition 2.1 in [6] that there is a recursive martingale d that succeeds on $A \oplus B$ with respect to a recursive bound f. Therefore, there are infinitely many n such that $d((A \oplus B)\upharpoonright 2(f(n)+1)) > 8^{n+1}$. We let S be the set of n where this holds.

Now we divide our proof into cases. Consider the set of strings of length $f(n)+1$. We ask whether there are infinitely many $n \in S$ such that

$$2^n < \frac{\sum_{|\sigma|=f(n)+1} d((A \oplus \sigma)\upharpoonright 2(f(n)+1))}{2^{f(n)+1}},$$

that is, whether the average value of d applied to $A \oplus \sigma$ for all σ of the appropriate length is greater than 2^n for infinitely many members of S.

If the answer is yes, then we have a contradiction, since this average value is also a recursive martingale that succeeds on A in the sense of Schnorr, and A is not Schnorr random.

If the answer is no, then

$$2^n \geq \frac{\sum_{|\sigma|=f(n)+1} d((A \oplus \sigma)\upharpoonright 2(f(n)+1))}{2^{f(n)+1}}$$

for all but finitely many $n \in S$. We define a new martingale m that is recursive in A based on a countable collection of martingales $\langle m_k \rangle_{k \in \omega}$ that is defined as follows.

We define the initial value of m_n to be $\frac{1}{2^{n+1}}$. Now, for each n, we let c_n be the number of strings σ of length $f(n)+1$ such that $d((A \oplus \sigma)\upharpoonright 2(f(n)+1)) \geq 8^{n+1}$. If σ is such a string, we define $m_n(\sigma) = \frac{2^{f(n)-n}}{c_n}$. If σ is a string of length $f(n)+1$ for which that inequality does not hold, we let $m_n(\sigma) = 0$. For all longer strings, we define m_n to be a nonbetting, constant martingale. Now we define $m(\sigma) = \sum_k m_k(\sigma)$ for all strings σ.

If $n \in S$ and n is not one of the finitely many exceptions, then

$$c_n \cdot 8^{n+1} \leq 2^{f(n)+1} 2^n,$$

so $c_n \leq 2^{f(n)-2n-2}$. Therefore, for almost all $n \in S$,

$$m_n(B\upharpoonright(f(n)+1)) \geq \frac{2^{f(n)-n}}{c_n} \geq 2^n,$$

and

$$m(B\upharpoonright(f(n)+1)) = \sum_k m_k(B\upharpoonright(f(n)+1)) \geq 2^n.$$

We have now produced a martingale recursive in A that succeeds on B in the sense of Schnorr, so again we have a contradiction. □

However, the other direction of van Lambalgen's Theorem does not hold for recursive randomness or Schnorr randomness. In [12], Merkle et al. showed that there is a recursively random real $A_0 \oplus A_1$ such that A_i is not Schnorr random relative to A_{1-i} for some $i \leq 1$. In addition, the following result and proof appear in [14].

Theorem 2.2 (Kjos-Hanssen)**.** *There is a Schnorr random real that does not satisfy van Lambalgen's Theorem.*

Proof. There is a high minimal degree by the Cooper Jump Inversion Theorem [2]. There must be a Schnorr random real $A_0 \oplus A_1$ in this high degree [15], and the only possible Turing degrees for A_0 and A_1 are $\mathbf{0}$ and that of $A_0 \oplus A_1$ itself. Since A_0 and A_1 are Schnorr random, they cannot be recursive, so $A_0 \equiv_T A_1 \equiv_T A_0 \oplus A_1$. Clearly, A_0 cannot be Schnorr random relative to A_1 (and vice versa), so van Lambalgen's Theorem does not hold for this particular Schnorr random real. □

We note that this proof can also be used to show that there is a recursively random real that does not satisfy van Lambalgen's Theorem, since every high degree also contains a recursively random real [15]. More recently, Yu has shown that van Lambalgen's Theorem fails with respect to Schnorr randomness and recursive randomness for a particular class of reals [22].

Theorem 2.3 (Theorem 0.9, [22]). *Let $B <_T 0'$ be an r.e. set. If $A = A_0 \oplus A_1 \leq_T B$ is recursively (Schnorr) random but not Martin-Löf random, then A_i is not A_{1-i}-recursively (Schnorr) random for $i \leq 1$.*

Kjos-Hanssen's and Yu's results only apply to very selective classes of reals: those in high minimal degrees and those bounded away from $\mathbf{0}'$ by an r.e. set. While all of the Martin-Löf reals in these classes are high, since every nonhigh Schnorr random real is Martin-Löf random [15], they are not very strong computationally. Either they are minimal, or they are computable in the halting problem. This leads to a very natural question: is there a high Turing degree $\mathbf{d}$ such that van Lambalgen's Theorem holds with respect to recursive randomness or Schnorr randomness for all random reals of the appropriate type in $\mathbf{d}$?

First, we consider reals $A \oplus B$ such that $A \oplus B$ is recursively (Schnorr) random and A and B are each recursively (Schnorr) random relative to each other. In Section 3, we show that there is at least one recursively random real $A \oplus B$ in every high degree for which van Lambalgen's Theorem fails with respect to recursive randomness and Schnorr randomness; that is, A is not recursively (Schnorr) random relative to B or vice versa. However, we go on to show in Section 4 that this failure is not universal in the high degrees; i.e., we show that van Lambalgen's Theorem does hold with respect to recursive randomness for some recursively random reals that are not Martin-Löf random. Finally, in Section 5, we will consider a recursively (Schnorr) random real A and attempt to describe the class of reals B that are recursively (Schnorr) random relative to A such that A is also recursively (Schnorr) random relative to B. This will help us characterize those reals B for which $A \oplus B$ is a recursively (Schnorr) random real that satisfies van Lambalgen's Theorem.

3. When van Lambalgen's Theorem fails

We will construct a real in an arbitrary high degree on which no recursive martingale succeeds. This real will be constructed in such a way that it can be recursively decomposed into two parts, one of which is not recursively random or Schnorr random relative to the other. We will need the following result, which follows easily from a theorem in [5].

Fact 3.1. If $\mathbf{a}$ is a high Turing degree, then there is an $A \in \mathbf{a}$ such that A *wtt*-computes a dominating function, that is, a function g such that for every computable function h, $g(n) > h(n)$ for all but finitely many n.

Theorem 3.2. *Let $\mathbf{a}$ be a high Turing degree. Then there is a $B \in \mathbf{a}$ such that B is recursively random and van Lambalgen's Theorem does not hold for B with respect to recursive randomness or Schnorr randomness.*

Proof. We let A be an element of $\mathbf{a}$ that *wtt*-computes a dominating function g. We must now produce a real B such that B is recursively random, $B \equiv_T A$, and B does not satisfy van Lambalgen's Theorem.

To satisfy the first of these conditions, we will use g to construct a martingale d that succeeds whenever any recursive martingale does and then construct the real B so that d cannot succeed on it. This will be enough to ensure that B is computable from A and that B is recursively random. However, we must also ensure that A is computable from B. To do this, we will construct B in segments of lengths determined by a strictly increasing recursive function f using a technique similar to that used in [11] (see Fact 3.3). These segments will increase in length and will code more and more of A. Finally, the bits of B at indices in the range of f will be computable from those that are not, so we will have an infinite, recursively identifiable portion of B that is computable from the rest of B. Observe that we are not decomposing B in the standard way into two reals C_0 and C_1 such that the $2n^{th}$ bit of B is the n^{th} bit of C_0 and the $(2n-1)^{st}$ bit of B is the n^{th} bit of C_1. Instead, we will decompose B into two reals B_0 and B_1 such that the n^{th} bit of B_0 is the $f(n-1)^{th}$ bit of B and the n^{th} bit of B_1 is the n^{th} bit of B that is not in the range of f. This will not matter since recursive randomness and Schnorr randomness are closed under recursive permutations. Therefore, van Lambalgen's Theorem will fail for B with respect to recursive and Schnorr randomness. While we could produce such a B_0 and B_1 such that B decomposes into B_0 and B_1 in the standard way (or, indeed, so the places in which B_1 are coded have arbitrarily low density), we prefer not to clutter the paper with these technical calculations.

At this point, we will not define f, and the reader should simply bear in mind that f is a recursive function.

We begin by describing a sequence of martingales $\langle d_k \rangle_{k\in\omega}$ that will contain all recursive martingales. We define each martingale d_k in terms of the following four functions: the k^{th} partial recursive function φ_k, a recursive bijection $r : \omega \to (0,2) \cap \mathbb{Q}$, and the dominating function g and recursive function f mentioned above.

For each k, we define d_k as follows. We let the initial capital of d_k be 1; i.e., d_k takes the value 1 on the empty string. Now suppose that we have defined d_k for a string σ such that $f(n) \leq |\sigma| < f(n+1)$. To define d_k on $\sigma 0$ and $\sigma 1$, we first determine whether $\varphi_k(\tau)$ converges within $g(n)$ steps for all τ such that $f(n) \leq |\tau| < f(n+1)$. If the answer is yes and $n > k$, we extend d_k as follows.

$$\begin{aligned} d_k(\sigma 0) &= r(\varphi_k(\sigma)) d_k(\sigma) \\ d_k(\sigma 1) &= (2 - r(\varphi_k(\sigma))) d_k(\sigma) \end{aligned}$$

Otherwise, we set $d_k(\sigma 0) = d_k(\sigma 1) = d_k(\sigma)$. Without loss of generality, we can define the initial value of each d_k to be 1.

Now we can define d as a weighted sum of the d_k by setting

$$d(\sigma) = \sum_k \frac{1}{2^{k+1}} d_k(\sigma)$$

for all $\sigma \in \{0,1\}^{<\omega}$. Note that d is recursive in A.

Now we define f. We begin by defining an auxiliary function u, where $u(n)$ is one plus the upper bound on the use of the calculation of $g(n+1)$ from A. Since $g \leq_{wtt} A$, we may take u to be recursive.

We will now make use of the following fact from [11].

Fact 3.3. Given a rational $\delta > 1$ and $k \in \omega$, we can recursively compute a length $l(\delta, k)$ such that for any martingale m and any $\sigma \in \{0,1\}^{<\omega}$, the following inequality holds.

$$|\{\tau \in \{0,1\}^{l(\delta,k)} \mid m(\sigma\tau) \leq \delta m(\sigma)\}| \geq k$$

This fact allows us to find a recursive function w such that for each σ of length $f(n)+1$, there are $2^{u(n)}$ strings τ of length $(f(n)+1)+w(n)$ extending σ such that

$$d(\tau) \leq \left(1+\frac{1}{2^n}\right) d(\sigma).$$

Now we can finally define f recursively by letting $f(0)=0$ and

$$f(n+1) = f(n)+1+w(n).$$

At this point, we are ready to construct B. Rather than define it bit by bit, we will define B on intervals: particularly, on intervals of the form $(f(n), f(n+1)]$. To do this, we will construct two sequences: $\langle \sigma_n \rangle_{n\in\omega}$ and $\langle \tau_n \rangle_{n\in\omega}$. For every $i<j$, we will have $\sigma_i \subseteq \sigma_j$, and the limits of the σ_is will be our desired real B. The τ_is will simply be auxiliary strings used to define the σ_is.

We let σ_0 and τ_0 both be the empty string. For each $n \geq 1$, we suppose that we are given a string σ_n in $\{0,1\}^{f(n)}$. We first extend it by either 0 or 1 to obtain a τ_n in $\{0,1\}^{f(n)+1}$ as follows.

$$\tau_n = \begin{cases} \sigma_n 0 & \text{if } d(\sigma_n 0) \leq d(\sigma_n 1) \\ \sigma_n 1 & \text{else} \end{cases}$$

We can now find a subset S_{n+1} of $\{0,1\}^{f(n+1)}$ such that each $\rho \in S_{n+1}$ extends τ_n and $d(\rho) \leq \left(1+\frac{1}{2^n}\right) d(\tau_n)$. We know from Fact 3.3 that, given our choice of f, there will be at least $2^{u(n)}$ elements in S_{n+1}. Therefore, we may code all binary strings of length $u(n)$ as one of the leftmost $2^{u(n)}$ elements of S_{n+1}. We now define σ_{n+1} to be the element of S_{n+1} that codes the first $u(n)$ bits of A.

Let $B = \lim \sigma_n$.

Lemma 3.4. *If the martingale d does not succeed on a real X, then no recursive martingale will succeed on X.*

Proof. Suppose that X is a real on which d does not succeed, and assume for a contradiction that there is a recursive martingale that succeeds on X. We note that if a recursive martingale that succeeds on X exists, then there is also a recursive martingale m that succeeds on X whose range consists solely of positive rationals [18]. Therefore, without loss of generality, we can simply consider m, which is definable as

$$\begin{aligned} m(\sigma 0) &= r(\varphi_k(\sigma))m(\sigma) \\ m(\sigma 1) &= (2-r(\varphi_k(\sigma)))m(\sigma) \end{aligned}$$

for some k, where r is the bijection from ω to $(0,2)\cap\mathbb{Q}$ defined above. However, when we defined each martingale d_k, we did not simply use the relationship above. Instead, we put constraints on the definition based on whether $\varphi_k(\sigma)$ converged for a certain set of σ within a certain number of steps determined by our function g. Since g is a dominating function, there are only finitely many places at which g does not bound the runtime of φ_k. Thus, up to a constant, the value that d_k takes on a string is no larger than that of m on the same string. Since d is simply the weighted sum of the d_k, it is clear that if d does not succeed on the real X, neither will d_k or m. This gives us a contradiction, so no recursive martingale can succeed on X. $\square$

We can see easily that d does not succeed on B since the value of d on B is bounded by $\prod_n(1+\frac{1}{2^n})$. This product converges, so by Lemma 3.4, no recursive martingale succeeds on B, and B is therefore recursively random.

Now we must show that $B \equiv_T A$. It is clear from the construction that $B \leq_T A$. To compute $A(n)$ from B, we begin by finding a k such that $n \leq u(k)$. Now we consider $B \upharpoonright f(k+1)$. We recall that we have coded the first $u(k)$ bits of A into the segment of B of bits with indices between $f(k)+1$ and $f(k+1)$. Only the first $u(k-1)$ bits of A and recursive functions were used in this coding, and this information can be obtained from a shorter initial segment of B (namely, $B \upharpoonright f(k)$), so we can see that we can recursively compute $A(n)$ from B for any n.

Finally, we must show that B does not satisfy van Lambalgen's Theorem with respect to recursive randomness or Schnorr randomness. As previously mentioned, we will not do this in the customary way by decomposing B into two reals, one consisting of its bits with even indices and one of its bits with odd indices. Instead, we will consider a different recursive decomposition of B. We use the following notation to simplify matters.

Notation 3.5. Let Z be an infinite recursive subset of ω. For any real X, we may write $X = X_0 \oplus_Z X_1$, where $X_0(n)$ is the bit of X whose index is the n^{th} element of Z and $X_1(n)$ is the bit of X whose index is the n^{th} element of $\overline{Z}$.

Let $F = \operatorname{ran}(f)$. Since f is a strictly increasing recursive function, F will be infinite and recursive. Therefore, we can express B as $B_0 \oplus_F B_1$. We can see that B_0 is recursive in B_1, since we can calculate $B(f(n))$ from $B \upharpoonright f(n)$ for all n. Therefore, B_0 is clearly not recursively random relative to B_1, and B does not satisfy van Lambalgen's Theorem with respect to recursive randomness. Furthermore, since f is recursive, we can recursively compute a lower bound on the amount a relativized recursive martingale can win by betting on the range of f, so B does not satisfy van Lambalgen's Theorem for Schnorr randomness, either. □

We take a moment here to mention the ways in which van Lambalgen's Theorem can fail for a real with respect to either recursive or Schnorr randomness in a very general sense. Suppose that we have a real $C = C_0 \oplus C_1$ such that C_0 is not recursively random relative to C_1, so there is a recursive martingale relative to C_1 that succeeds on C_0. There are two possible reasons that this martingale could not be converted to a recursive martingale that succeeds on C: either its use is too large to read all the necessary bits of C_1 before betting on a bit of C_0, or it is not always defined on $C_0 \oplus D$ for every real D. However, Miyabe has shown that for a certain formulation of "truth-table Schnorr randomness," van Lambalgen's Theorem holds [13].

We now briefly consider van Lambalgen's Theorem in the context of Kurtz randomness. Kurtz randomness was first defined in [9] and is a weaker notion than either Schnorr or recursive randomness. Recall that the hyperimmune Turing degrees are precisely those that compute a function that is not majorized by any recursive function.

Definition 3.6. A real A is Kurtz random if $A \in U$ for every r.e. open set U of measure 1.

We first observe that one direction of van Lambalgen's Theorem holds with respect to Kurtz randomness.

Theorem 3.7. *If A_0 is Kurtz random and A_1 is Kurtz random relative to A_0, then $A_0 \oplus A_1$ is Kurtz random.*

Proof. We assume that A_0 is Kurtz random and let U be an arbitrary r.e. open set of measure 1. Now, for a rational $r < 1$, we define the class $U_r = \{P \mid \mu(\{Q \mid P \oplus Q \in U\}) > r\}$. Each U_r can be enumerated recursively, so each U_r is an r.e. open set. Furthermore, $\mu(U_r) = 1$ for each r since otherwise the measure of U would be strictly less than 1. Since A_0 is Kurtz random, $A_0 \in U_r$

for each r, and therefore the set $T = \{Q \mid A_0 \oplus Q \in U\}$ has measure 1 and is r.e. relative to A_0. Since A_1 is assumed to be Kurtz random relative to A_0, A_1 must be an element of T and therefore $A_0 \oplus A_1$ must be an element of our arbitrary r.e. open set U. □

We now show that van Lambalgen's Theorem fails with respect to Kurtz randomness in a larger class of Turing degrees than Schnorr or recursive randomness.

Theorem 3.8. *Every hyperimmune Turing degree contains a Kurtz random real that does not satisfy van Lambalgen's Theorem with respect to Kurtz randomness.*

Proof. Let E be an element of a hyperimmune Turing degree, and let $f \leq_T E$ be such that f is not majorized by any recursive function; that is, there is no recursive function g such that $g(n) \geq f(n)$ for all n. We will build a real $A \oplus B$ by finite extensions such that $A \oplus B$ is Kurtz random and neither A nor B is Kurtz random relative to the other.

We begin by dividing the natural numbers into a sequence of intervals $\langle I_k \rangle_{k \in \omega}$ such that $I_k = \{2^k, 2^k + 1, \ldots, 2^{k+1} - 1\}$ and say that an extension function is a partial recursive function that, given a finite binary string σ, will output a finite binary string τ such that the length of τ is 2^k for some k and τ extends σ. To ensure that the real we build is Kurtz random, we will require that it meet all extension functions φ_e that are not constant on either the even half (the bits with even indices) or on the odd half (the bits with odd indices) of any of these intervals. This will be enough, since any extension function that is constant on either half of one of these intervals will produce an r.e. open set with measure strictly less than 1.

We let σ_0 be the string with domain I_0 that takes the value $E(0)$ on all bits with indices in I_0. At stage $n \geq 1$, we assume that we have defined σ_{n-1} and that its domain is the union of the intervals $I_0, I_1, \ldots, I_m$ for some m. Now we list those $e \leq n$ such that the computation of $\varphi_e(\sigma_{n-1})$ converges within $f(n)$ steps and let e' be the least index for which this computation converges in the appropriate time and for which $\varphi_{e'}$ has not already been utilized at a previous stage. If such an e' exists, we set $\tau = \varphi_{e'}(\sigma_{n-1})$; otherwise, we let $\tau = \sigma_{n-1}$. Suppose that τ is defined on the intervals $I_0, I_1, \ldots, I_k$. Now we define σ_n to be the string that extends τ to the interval I_{k+1} by giving it the value $E(n)$ everywhere on this interval.

Let $A \oplus B = \lim_n \sigma_n$. It is clear that $A \oplus B \leq_T E$. We can see that $A \oplus B$ meets all extension functions of the appropriate type, since f is not majorized by any recursive function, so $A \oplus B$ is Kurtz random. Furthermore, we can calculate E from A. Since the even bits of $A \oplus B$ are only constant on the intervals that code the bits of E, all we need to do to find the k^{th} bit of E is to consider the intervals of A that correspond to intervals I_m of $A \oplus B$ and have constant values. The only value A will take on the k^{th} such interval is $E(k-1)$, so $E \leq_T A$. Similarly, we can see that $E \leq_T B$, so $A \oplus B \equiv_T E \equiv_T A \equiv_T B$, and A and B cannot be Kurtz random relative to each other. □

We note that since Kurtz randomness coincides with Martin-Löf randomness in the hyperimmune-free Turing degrees [15], this gives us a full characterization of the Turing degrees that contain Kurtz random reals for which van Lambalgen's Theorem fails with respect to Kurtz randomness.

4. When van Lambalgen's Theorem holds

We have seen that every high degree contains a recursively random (and therefore Schnorr random) real for which van Lambalgen's Theorem fails with respect to recursive randomness and

Schnorr randomness. One obvious question remains: is there any recursively random or Schnorr random real that is not Martin-Löf random for which van Lambalgen's Theorem holds with respect to either of these notions? In this section, we answer this question positively. We further produce an example of such a Schnorr random real.

Theorem 4.1. *There is a recursively random real that is not Martin-Löf random and satisfies van Lambalgen's Theorem for recursive randomness.*

Proof. In [7], a recursively random real R in a high but incomplete r.e. Turing degree was constructed that is recursively random relative to all elements of a class of measure 1. The existence of such a real can also be deduced easily from Theorem 1.2 in [1]. This real is clearly not Martin-Löf random since any Martin-Löf random real that has r.e. Turing degree must be Turing complete [8]. Now we observe that the class of 2-random reals has measure 1, so there is a 2-random real A such that R is recursively random relative to A. Furthermore, since A is 2-random, A is Martin-Löf random relative to R.

Since both reals are recursively random relative to each other, $A \oplus R$ is recursively random by Theorem 2.1. However, since R is not Martin-Löf random, $A \oplus R$ is not either. □

We also outline a construction of a particular Schnorr random real that is not Martin-Löf random. This construction involves Ω, the halting probability of a universal prefix-free Turing machine.

Example 4.2. To construct a Schnorr random real that is not Martin-Löf random for which van Lambalgen's Theorem holds, we will first define two reals based on Ω that are Schnorr random with respect to each other. We begin by fixing an approximation $\langle \Omega_s \rangle_{s\in\omega}$ to Ω. We also use the convergence modulus of Ω, the function defined as $c_\Omega(n) = \min\{s \mid \Omega_s \restriction n = \Omega \restriction n\}$, to define the sequence $\langle a_n \rangle_{n\in\omega}$, where $a_0 = 0$ and $a_{n+1} = c_\Omega(a_n) + 1$ for all n. We use this sequence to partition the natural numbers into infinitely many intervals $\langle I_n \rangle_{n\in\omega}$, where $|I_0| = 1$, $|I_{2k+1}| = 1$, and $|I_{2k+2}| = a_{k+1} - a_k$ for all k.

We can now define a real A that is equal to 0 on odd intervals, $\Omega(0)$ on I_0, and $\Omega(a_k+1)\Omega(a_k+2)\ldots\Omega(a_{k+1})$ on I_{2k+2}. Note that $A \leq_T \Omega$. Now consider Ω^Ω, which is the halting probability relative to Ω. This real is Martin-Löf random and thus Schnorr random relative to Ω, and since $A \leq_T \Omega$, it is also Schnorr random relative to A. Furthermore, we can argue that A is Schnorr random relative to Ω^Ω. To do so, we note that A is simply a variant of Ω: to create A, we distributed the bits of Ω just sparsely enough that no useful information can be obtained about them in recursive time, even using Ω^Ω. Therefore, $A \oplus \Omega^\Omega$ is Schnorr random by Theorem 2.1.

However, A is not Martin-Löf random relative to Ω^Ω, since Ω is Martin-Löf random relative to Ω^Ω and c_Ω dominates every Ω^Ω-recursive function. Van Lambalgen's Theorem tells us that $A \oplus \Omega^\Omega$ is not Martin-Löf random, so we have an example of a Schnorr random real that is not Martin-Löf random that satisfies van Lambalgen's Theorem with respect to Schnorr randomness.

5. Another approach

As mentioned before, there are two primary ways in which we may consider a random real X in the context of van Lambalgen's Theorem. First, we may ask, as we have above, whether X can be recursively decomposed into an X_0 and X_1 such that one X_i is not random with respect to the other. However, we may also ask whether X is half of a real for which van Lambalgen's Theorem fails, i.e., whether there is another real Y such that $X \oplus Y$ is random but van Lambalgen's Theorem fails for $X \oplus Y$. It turns out that every Schnorr random real is half of another Schnorr random real for which van Lambalgen's Theorem fails with respect to Schnorr randomness.

Theorem 5.1. *If A is a Schnorr random real, then there is a real B such that $A \oplus B$ is Schnorr random and van Lambalgen's Theorem fails for $A \oplus B$ with respect to Schnorr randomness.*

We will need the following result to prove this theorem. Recall that a maximal set is an r.e. set E that has an infinite complement for which there is no r.e. set W such that $|W \cap \overline{E}| = |\overline{W} \cap \overline{E}| = \infty$.

Proposition 5.2. [3] *If R is Schnorr random and E is maximal, then $R \cap E$ is Schnorr random.*

Proof of Theorem 5.1. We consider the case in which A Turing computes the halting set K and the case in which A does not Turing compute K separately.

If A is Schnorr random and $A \not\geq_T K$, by Theorem 2.1 in [7], there is a real $B \geq_T A$ such that B is Schnorr random relative to A precisely when $A \not\geq_T K$. Since A is Schnorr random, by our Theorem 2.1, $A \oplus B$ must be Schnorr random. However, since A is not Schnorr random relative to B, van Lambalgen's Theorem fails for $A \oplus B$ with respect to Schnorr randomness.

If A is Schnorr random and $A \geq_T K$, we let R be Martin-Löf random with respect to A and consider $B = R \cap E$ for some maximal set E. We note that $A \oplus R$ is Schnorr random by Theorem 2.1 and that $\omega \oplus E$ is maximal. By Proposition 5.2, we can see that $A \oplus B = (A \oplus R) \cap (\omega \oplus E)$ is Schnorr random. However, since $E \leq_T A$, we can create a martingale recursive in A that succeeds on B by computing E and then betting that $B(n) = 0$ whenever n is not in E. We can also find an order function recursive in A that grows sufficiently slowly that this martingale succeeds at the rate indicated by the order function, so we can see that B is not Schnorr random relative to A and, once again, van Lambalgen's Theorem fails for $A \oplus B$ with respect to Schnorr randomness. □

We may also ask the following question for those Schnorr random reals which Turing compute K.

Question 5.3. Suppose that $A \geq_T K$ is Schnorr random. Must there be a real $B \leq_T A$ such that B is Schnorr random and A is Schnorr random relative to B?

A positive answer to this question would not only refine the second part of the proof of Theorem 5.1 but also provide a sort of randomness inversion theorem.

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Department of Mathematics, 6188 Kemeny Hall, Dartmouth College, Hanover, NH 03755-3551, USA
E-mail address: `johannaf@gauss.dartmouth.edu`

Department of Computer Science and Department of Mathematics, National University of Singapore, Singapore 117543, Singapore
E-mail address: `fstephan@comp.nus.edu.sg`