SUBCLASSES OF THE WEAKLY RANDOM REALS

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ABSTRACT. The weakly random reals contain not only the Schnorr random reals as a subclass but also the weakly 1-generic reals and therefore the n-generic reals for every n. While the class of Schnorr random reals does not overlap with any of these classes of generic reals, their degrees may. In this paper, we describe the extent to which this is possible for the Turing, weak truth-table, and truth-table degrees and then extend our analysis to the Schnorr random and hyperimmune reals.

1. INTRODUCTION

Randomness and genericity are somehow similar concepts. A real that is random is, in some sense, large with respect to measure, and a real that is generic may be considered to be large with respect to category. The degree to which random reals and generic reals may be related is, therefore, of interest. Given a very weak notion of randomness, a real may be both random and generic, but for any reasonably strong definition of randomness, this is not the case.

Weak randomness, developed by Kurtz in his thesis [11] and therefore also called Kurtz randomness, is the weakest of all the commonly discussed randomness notions. Not only are all the reals that are Schnorr random weakly random, but so are all the reals that are weakly 1-generic. This implies that for every n, all the reals that are n-random or n-generic are weakly random. However, the n-random reals and the m-generic reals do not overlap for any n and m.

In this paper, we study the relationships between the degrees of these subclasses of the weakly random reals. The first part of the paper consists of an analysis of the relationship between the degrees of random reals and the degrees of generic reals. In the second part of this paper, we generalize the genericity condition to that of hyperimmunity and consider the relationship between the degrees of random reals and hyperimmune reals.

1.1. **Background.** Our notation generally follows that of Soare [17] and Odifreddi [14, 15]. We work within the Cantor space, denoted by 2^ω, and we call its elements reals. We will use μ to denote the Lebesgue measure on 2^ω throughout. For a finite binary string σ and a finite or infinite binary string C, we write $\sigma \subseteq C$ to indicate that σ is an initial segment of C. Although it may be more typical to denote this relationship by $\preceq$, we will use this notation in Section 2 to indicate the ordering we place on a set of forcing conditions instead, as is standard in set theory. Furthermore, for a finite binary string σ, we let $[\sigma]$ denote the class of reals extending σ: $\{A \mid \sigma \subseteq A\}$.

The original definition of weak randomness is unlike the standard definition of most other randomness notions. Generally, a real is considered to be random if it avoids all null sets defined in some particular effective way. Kurtz proposed in his thesis that a real could be considered to be random if, instead of avoiding every null set, it is contained in every effectively defined set of measure 1 [11].

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Definition 1.1. [11] A real A is weakly random if $A \in U$ for every Σ^0_1 set $U \subseteq 2^\omega$ of measure 1.

Wang proved that this class of reals can also be defined in the same way that Schnorr and Martin-Löf randomness typically are; e.g., in terms of tests.

Definition 1.2. [19] A Kurtz null test is a sequence $\langle V_n \rangle_{n\in\omega}$ of open subsets of the Cantor space such that for every n, $\mu(V_n) \leq \frac{1}{2^n}$ and $V_n = \bigcup_{\sigma \in f(n)}[\sigma]$ for a given recursive function $f : \omega \to (2^{<\omega})^{<\omega}$.

Theorem 1.3. [19] *A real A is weakly random if and only if for every Kurtz null test $\langle V_n \rangle_{n\in\omega}$, $A \notin \bigcap_{n\in\omega} V_n$.*

We now describe two stronger notions of randomness: Martin-Löf randomness and Schnorr randomness. For Martin-Löf randomness, we increase the class of tests whose null sets every random real must avoid by allowing each V_n to be determined by an infinite r.e. set instead of a finite set. When we consider Schnorr randomness, we use Martin-Löf tests that are restricted with respect to measure.

Definition 1.4. [12] A Martin-Löf test is a sequence $\langle V_n \rangle_{n\in\omega}$ of open subsets of the Cantor space such that for every n, $\mu(V_n) \leq \frac{1}{2^n}$ for every n and $V_n = [W_{f(n)}]$ for a given recursive function f. A real A is said to be Martin-Löf random if for every Martin-Löf test $\langle V_n \rangle_{n\in\omega}$, $A \notin \bigcap_{n\in\omega} V_n$.

Definition 1.5. [16] A Martin-Löf test $\langle V_n \rangle_{n\in\omega}$ is said to be a Schnorr test if for every n, $\mu(V_n) = \frac{1}{2^n}$. A real A is said to be Schnorr random if for every Schnorr test $\langle V_n \rangle_{n\in\omega}$, $A \notin \bigcap_{n\in\omega} V_n$.

It is clear that every Martin-Löf random real is Schnorr random. It can also be shown that every Schnorr random real is weakly random. The proof of this result involves the characterizations of these randomness notions based on unpredictability. We use martingales to formalize these characterizations. Recall that a martingale d is simply a function from $2^{<\omega}$ to $\mathbb{R}^{\geq 0}$ such that for every string σ, $d(\sigma) = \frac{d(\sigma 0)+d(\sigma 1)}{2}$, and that a martingale d is r.e. (recursive) if the values $d(\sigma)$ are uniformly r.e. (recursive) reals.

Theorem 1.6. [16, 19] *Suppose A is a real.*

1. *A is Schnorr random if there is no recursive martingale d such that $d(A\restriction n) \geq h(n)$ for infinitely many n for some unbounded, nondecreasing recursive function h.*
2. *A is weakly random if there is no recursive martingale d and no unbounded, nondecreasing recursive function h such that for some n, $d(A\restriction n) \leq h(n)$.*

It is clear from this theorem that every Schnorr random real (and thus every Martin-Löf random real) is weakly random.

Finally, we mention a form of randomness strictly intermediate between Martin-Löf randomness and Schnorr randomness: recursive randomness. It is most naturally characterized in terms of martingales.

Definition 1.7. [16] A real A is recursively random if there is no recursive martingale d that succeeds on A; i.e., such that $\limsup_n d(A\restriction n) = \infty$.

We now note that although weak randomness is primarily considered a randomness notion, it would not be inappropriate to consider it as a genericity notion since every weakly random real meets every sufficiently large Σ^0_1 set. When we discuss genericity, we will use the formulations in [11].

Definition 1.8. A real G forces a statement φ if there is some initial segment σ of G such that φ is true of all extensions of σ.

A set $S \subseteq 2^{<\omega}$ is said to be dense if for every $\sigma \in 2^{<\omega}$, there is some $\tau \in S$ such that $\sigma \subseteq \tau$.

Definition 1.9. A real G is n-generic if for every Σ^0_n sentence φ, either G forces φ or G forces $\neg\varphi$, and a real G is weakly n-generic if for every dense Σ^0_n set S, there is some $\sigma \in S$ such that $\sigma \subset G$.

It can be seen from this definition that every real that is weakly 1-generic is also weakly random. Furthermore, every n-generic real is weakly n-generic, and every weakly $(n+1)$-generic real is n-generic [11].

We will also consider hyperimmunity, a more general notion than genericity.

Definition 1.10. A real A is hyperimmune if A is infinite and no recursive function dominates p_A, the function that lists those n such that $A(n) = 1$ in increasing order.

1.2. **Previous work.** We first note that no real can be both Schnorr random and weakly 1-generic [2]. To see this, we construct a dense r.e. set of strings $S = \cup_i S_i$ such that $\langle [S_i] \rangle_{i\in\omega}$ is a nested Schnorr test and any weakly 1-generic real must be contained in $[S_i]$ for infinitely many i. Any weakly 1-generic real will be an element of the intersection of the $[S_i]$s, so it cannot be Schnorr random.

Quite a lot of work has been done on the relationship between randomness and genericity. Demuth and Kučera proved in [1] that no 1-generic real Turing computes a Martin-Löf random real. This implies that no 2-generic real computes a 2-random real. In [13], Nies, Stephan, and Terwijn proved that, in fact, any 2-generic real and any 2-random real form a minimal pair and noted that this result cannot be improved. Since every real is weak truth-table computed by a Martin-Löf random real [9, 5], no 2-generic real can form a minimal pair with every Martin-Löf random real. Furthermore, every 2-random real Turing computes a 1-generic real [7].

Every weakly 1-generic real is hyperimmune [11], but not every weakly random real is, since there are weakly random reals that are hyperimmune-free [13]. Therefore, we may also consider the relationship between the Schnorr random reals and the weakly random hyperimmune reals.

2. Genericity and Schnorr randomness

It is clearly possible for a Schnorr random real and a 1-generic real to share a Turing degree: each high Turing degree contains a Schnorr random real [13], and there is a high 1-generic real. However, we can see that this highness condition is necessary and that, in fact, a nonhigh 1-generic real cannot even compute a Schnorr random real.

Theorem 2.1. *If a 1-generic real is not high, it cannot Turing compute a Schnorr random real.*

Proof. Let G be a 1-generic real that is not high, and suppose that it computes a Schnorr random real A. Since A is not high, it must be Martin-Löf random [13]. Every Martin-Löf random real is fixed-point free (FPF) [10], and the Turing degrees with FPF reals are closed upwards. Therefore, the Turing degree of G must be FPF as well, and since fixed-point freeness is degree invariant, G must be FPF as well. However, no 1-generic degree can be FPF [1], so we have a contradiction. $\square$

Since every 2-generic real is 1-generic and no 2-generic real is high, no 2-generic real Turing computes a Schnorr random real. We present a direct argument here that is similar to those in [4]. The interested reader may wish to compare the proof that Cohen forcing does not add a random real, which can be seen as an immediate corollary to Solovay's characterization of random reals in [18].

In this proof, we will make use of the machine characterization of Schnorr randomness, originally given by Downey and Griffiths [3]. We may consider a Turing machine M to be a partial recursive function from $2^{<\omega}$ to $2^{<\omega}$. A Turing machine is said to be prefix-free if there are no σ and τ in its domain such that σ extends τ. Finally, a Turing machine is said to be computable if the Lebesgue measure of its domain is a recursive real; i.e., effectively approximable from above as well as from below. The Kolmogorov complexity of a finite binary string σ with respect to a particular prefix-free Turing machine M is defined to be $K_M(\sigma) = \min\{|\tau| \mid K_M(\tau) = \sigma\}$.

Theorem 2.2. [3] *A real A is Schnorr random if for every prefix-free computable Turing machine M,*

$$(\exists c \in \omega)(\forall n \in \omega)[K_M(A \upharpoonright n) \geq n - c].$$

We will also need to make use of the Kraft-Chaitin Theorem.

Theorem 2.3 (Kraft-Chaitin Theorem)**.** *Let $\langle d_i, \sigma_i \rangle_{i \in \omega}$ be a recursive sequence with $d_i \in \omega$ and $\sigma_i \in 2^{<\omega}$ for all i such that $\sum_i \frac{1}{2^{d_i}} \leq 1$. (Such a sequence is called a Kraft-Chaitin set, and each element of the sequence is called a Kraft-Chaitin axiom.) Then there are strings τ_i and a prefix-free machine M such that $dom(M) = \{\tau_i \mid i \in \omega\}$ and for all i and j in ω,*

1. *if $i \neq j$, then $\tau_i \neq \tau_j$,*
2. *$|\tau_i| = d_i$,*
3. *and $M(\tau_i) = \sigma_i$.*

This theorem allows us to construct a prefix-free machine by specifying only the lengths of the strings in the domain rather than the actual strings. This allows us to identify $\langle \tau, \sigma \rangle$ with $\langle d, \sigma \rangle$, where $d = |\tau|$, throughout.

Theorem 2.4. *Suppose G is 2-generic and $A \leq_T G$. Then A cannot be Schnorr random.*

Proof. Let G be 2-generic, and let Ψ be a Turing function witnessing $A \leq_T G$. Given an oracle X, the statement that Ψ^X is total can be written as follows.

$$\varphi^X = (\forall n \in \omega)(\exists s \in \omega)[\Psi_s^X(n) \downarrow]$$

Since φ^X is a $\Pi_2^{0,X}$ statement, G must either force φ^G to be true or force it to be false. Since $A \leq_T G$, G cannot force it to be false, so G must force its truth. Call the initial segment that does so p. Our forcing conditions will be the set $\mathcal{P} = \{q \in 2^{<\omega} \mid q \supseteq p\}$, which we can recursively enumerate as $\langle q_i \rangle_{i \in \omega}$. We follow the standard convention of ordering $\mathcal{P}$ by writing $q_i \preceq q_j$ when $q_i \supseteq q_j$.

We may now consider the set $T = \{r_i \mid r_i = \Psi^{q_i}\}$. Note that there may be some i and j for which $r_i = r_j$. Since p forces the totality of Ψ, for every element r_i of T, there will be some r_j in T extending r_i. Therefore, we can think of the elements of T as an infinite r.e. binary tree, and A will be one of the paths through T. To prove that A is not Schnorr random, we will show that we cannot force A to be Schnorr random.

To do this, we will build a computable Turing machine M such that for each constant c and each r_i, there is an extension p_i of r_i such that $K_M(p_i) < n - c$. This will guarantee that for each pair c and i, we cannot force $K_M(p_i) \geq n - c$; that is, we will never be able to force Schnorr randomness above any r_i in our tree. To this end, we let $\langle \cdot, \cdot \rangle$ be a recursive bijection from $\omega \times \omega$ to $\omega - \{0\}$.

Our construction proceeds in stages. At stage 0, we set $M = \emptyset$. At stage $\langle c, i\rangle$, we choose $n \in \omega$ such that n is larger than all such n used at previous stages and such that $\langle c, i\rangle < n - c$. We enumerate the elements of T until we find some $r_j \supseteq r_i$ such that $|r_j| \geq n$ and then enumerate the Kraft-Chaitin axiom $\langle\langle c, i\rangle, r_j\restriction n\rangle$ into M.

At each stage $s > 0$, we added $\frac{1}{2^s}$ to the measure of dom(M), so $\mu(\text{dom}(M)) = \sum_{s>0} \frac{1}{2^s} = 1$. Therefore, we can apply the Kraft-Chaitin Theorem, and we can clearly think of M as not just a prefix-free Turing machine but a computable one.

All that remains to be shown is that A cannot be Schnorr random. To show that M witnesses that A is not Schnorr random, we consider the following statement in a real X and a constant c.

$$\psi^X(c) = (\exists n \in \omega)(\exists s \in \omega)[K_{M_s}(\Psi^X\restriction n) < n - c]$$

Since this is a $\Sigma_1^{0,X}$ statement in c and G is 2-generic, G must either force $\psi^G(c)$ or its negation for every constant c. We have constructed our machine M so that for every r_i and c, some extension of r_i has a complexity less than its length minus c. Since every initial segment of G is extended by some r_i, G must force the truth of this statement for every c, and $\Psi^G = A$ must therefore not be Schnorr random. □

We observe that we only use the full strength of the genericity of G when we force the totality of Ψ. When we force nonrandomness, we use only a Σ_1 statement and not a Π_2 statement. If we consider only truth-table functionals, we do not have to force the statement that Ψ is total. This allows us to weaken our assumptions about the generic and let it be simply 1-generic instead of 2-generic, giving us the following theorem.

Theorem 2.5. *Suppose G is 1-generic and $A \leq_{tt} G$. Then A cannot be Schnorr random.*

We may also ask if we can weaken the reducibility in Theorem 2.5 to weak truth-table reducibility. This turns out not to be possible. In particular, we show that this is impossible if we add an additional assumption concerning the degree of the 1-generic real; namely, the assumption that the real is high. However, if we make this assumption, it enables us to consider a weaker property than 1-genericity, resulting in an entirely different sort of proof. In particular, we assume that the real is GL_1 instead of 1-generic. Recall that a real B is GL_1 if and only if $B' \equiv_T B \oplus 0'$.

Theorem 2.6. *Suppose G is high and GL_1. Then there is a recursively random (and thus Schnorr random) real A such that $A \equiv_{wtt} G$.*

Since every real that is 1-generic is GL_1 [6], this will give us the following corollary immediately.

Corollary 2.7. *Suppose G is 1-generic and high. Then there is a recursively random (and thus Schnorr random) real A such that $A \equiv_{wtt} G$.*

Proof of Theorem 2.6. Since G is high, we know that $G' \equiv_T 0''$, and, since G is GL_1, we know that $G' \equiv_T G \oplus 0'$. This means that $0'' \equiv_T G \oplus 0'$, so $G \oplus 0'$ can determine whether a given r.e. martingale d is total. We will begin by creating a list recursively in G that will allow us to build a list of total martingales to use in our proof. While some martingales may be repeated, every total martingale will appear in the list at some point. To do this, we fix an enumeration $\langle d_e\rangle_{e\in\omega}$ of all r.e. martingales and let Φ be a Turing functional such that $\Phi^{G\oplus 0'}(e)$ equals 1 if d_e is total and 0 if it is not, and we fix an enumeration $\langle 0'_s\rangle_{s\in\omega}$ of $0'$. Without loss of generality, we assume that d_0 is the martingale such that $d_0(\sigma) = 1$ for every $\sigma \in 2^{<\omega}$; that is, the martingale that does not bet on anything, and we further assume that Φ does not consult the oracle at all when it determines whether d_0 is total. We use this to construct a list of elements of $\omega \times \omega \times \omega$ that we will use to identify a collection of total r.e. martingales that contains every total r.e. martingale. The first element of the triple will be the index e of an r.e., possibly total, martingale, the second will be the stage s at which the calculation of $\Psi_s^{G\oplus 0'_s}(e)$ indicates that we should add e to the list, and the third will be the use u of the approximation $0'_s$ in this calculation.

At stage 0, we add the triple $(0,0,0)$ to the list. If $s > 0$, we consider all $e \leq s$. If $\Psi_s^{G\oplus 0'_s}(e) = 1$ and the use of the $0'_s$ component is u, we add the triple (e, s, u) to the list. Whenever necessary, we will add the triple $(0,0,0)$ to the list to ensure that the k^{th} entry in the list (e_k, s_k, u_k) can be determined using only $G{\upharpoonright}k$ and that $e_k < k$ for every $k > 0$. This will ensure that our list is not only Turing computable from G but also *wtt*-computable from G. We may assume without loss of generality that each martingale d_{e_k} assumes only nonnegative rational values.

It should be observed at this point that even if (e, s, u) is on our list, the martingale d_e may not actually be total. It is possible that after the stage s at which we added (e, s, u), the approximation to $0'$ changed. If, for some $t > s$, $0'_t{\upharpoonright}u \neq 0'_s{\upharpoonright}u$, the computation $\Psi^{G\oplus 0'_t}(e)$ may not terminate in t steps or, if it does, it may even yield an answer of 0. Therefore, we will consider the approximation $0'_s$ at stage s in our computations below. If we are using d_e in our calculations because (e, s, u) is on our list and we find that $0'_t{\upharpoonright}u \neq 0'_s{\upharpoonright}u$ for some $t > s$, we will stop calculating additional values for d_e at this point. Instead, we will use the values we have calculated up to this point and treat d_e as a nonbetting martingale when we need any more of its values for a computation to ensure that we are using a total martingale. Of course, if d_e is a total martingale, (e, s, u) will be on our list for some s and u for which $0'_s{\upharpoonright}u = 0'{\upharpoonright}u$ and $0'_t{\upharpoonright}u = 0'_s{\upharpoonright}u$ for all $t \geq s$. After stage s, we will never find any evidence that d_e may not be total, so this entry in our list will result in d_e being used in our computation in its entirety. Therefore, our list of triples $\langle (e_k, s_k, u_k)\rangle_{k\in\omega}$ that is *wtt*-computable from G will allow us to develop a list of recursive martingales based on the sequence of r.e. martingles $\langle d_{e_k}\rangle_{k\in\omega}$ that we will use throughout the proof, and this list will still be *wtt*-computable from G. We will still refer to the k^{th} element of this sequence as d_{e_k} for the sake of simplicity, although the recursive martingale in question may only be based on the actual d_{e_k} and become nonbetting at some point.

We will also alter the martingales d_{e_k} slightly in another way. We choose a G-recursive partition $\langle I_k\rangle_{k\in\omega}$ of ω such that for every k, $\max(I_k) > k$ and there are $2(k+2)$ strings on I_k such that the first k martingales in our list grow by a factor of no more than $1 + \frac{1}{2^k}$ on each of them. Note that, in fact, the sequence $\langle I_k\rangle_{k\in\omega}$ is weak truth-table computable from G. For each k, this allows us to define a new martingale m_{e_k} based on d_{e_k} such that the following conditions hold.

(1) For all σ of length $< \max(I_k)$, $m_{e_k}(\sigma) = 1$.
(2) For all σ of length $\max(I_k)$ and all τ, $m_{e_k}(\sigma\tau) = \frac{1+d_{e_k}(\sigma\tau)}{1+d_{e_k}(\sigma)}$.

It is clear that if d_{e_k} succeeds on a real, so will m_{e_k}. Now we define a new martingale that is a weighted sum of the m_{e_k}s: for each σ in I_k, $m(\sigma) = \frac{1}{2^k}\sum_{i<k}\frac{1}{2^{i+1}}m_{e_i}(\sigma)$. Observe that m is a rational-valued martingale, since only finitely many of the m_{e_k}s are used to determine the value of $m(\sigma)$ for any given σ. It is also clear that $m \leq_{wtt} G$ since all of the m_{e_k}s and I_ks are weak truth-table computable from G. Furthermore, if any m_{e_k} succeeds on a real, so will m.

Now we use m to construct our real A by finite extensions. For each k, we define A on I_k as follows. We chose I_k so that there are at least $2(k+2)$ strings in this interval on which no martingale m_{e_i} such that $i < k$ grows by a factor larger than $1 + \frac{1}{2^k}$. We have chosen these intervals to be sufficiently long that we can use $A\restriction\max(I_{k-1})$ to find the values m assumes on I_k, so we can identify these strings from $A\restriction\max(I_{k-1})$. Then we choose the leftmost $2(k+2)$ such strings, order them lexicographically, and define A on I_k to be the string that is $(G(k)(k+2)+e_{k+1})^{st}$ in this set. This will not only allow us to compute $G(k)$ from A, it will let us determine the values of m on the next interval, I_{k+1}.

We first show that $G \equiv_{wtt} A$. We have computed A from G by partitioning ω into intervals I_k and using $G(k)$, the values of m_{e_i} for $i<k$, and an index $e_{k+1}<k$ to determine the values of A on I_k. Each of these computations is weak truth-table in G, so we can see that $A \leq_{wtt} G$. Furthermore, we can compute $G(k)$ from $A\restriction\max(I_k)$, and since $G\restriction k$ allows us to determine I_{k+1}, we know that $G \leq_{wtt} A$.

Finally, since the value of m increases by no more than a factor of $1+\frac{1}{2^k}$ on the interval I_k, the values of m on A will be bounded by $\prod_k(1+\frac{1}{2^k})$, which is convergent. Therefore, none of the martingales m_{e_k} succeed on A, and A must be recursively random and therefore Schnorr random. □

In this proof, we have used the highness of A and the fact that G is GL_1 to build a martingale that, while not a recursive martingale itself, covers all recursive martingales. This has allowed us to prove a stronger statement than desired: not only can we find a Schnorr random real that is weak truth-table equivalent to our G, we can find a recursively random real with this property.

We may also ask if we can strengthen the genericity condition in the previous corollary to weak 2-genericity. This turns out to be possible if the weakly 2-generic real is high. However, since no 2-generic real is high, we must first prove that such a real exists.

Theorem 2.8. *There is a high weakly 2-generic real.*

Proof. Here, we treat partial recursive functions as functions from $2^{<\omega}$ to $2^{<\omega}$ rather than ω to ω. Let an extension function be a partial recursive function φ such that for every σ, $\varphi(\sigma)$ extends σ, and define E to be the set of all indices of total extension functions that are recursive in $0'$.

We define our real G by finite extensions. At stage 0, we define σ_0 to be $0^{e_0}1$. Given σ_k, we define σ_{k+1} to be $\varphi^{0'}_{e_k}(\sigma_k)0^{e_{k+1}}1$. Let $G=\lim_k \sigma_k$.

Since G meets every extension function that is recursive in $0'$, G is weakly 2-generic. Furthermore, since the indices $\langle e_k\rangle_{k\in\omega}$ can be found recursively in G and $0'$, we can see that the function mapping k to σ_k is recursive in $G\oplus 0'$. Therefore, $E\leq_T G\oplus 0'$. We can use the *s-m-n* Theorem to produce a recursive function f such that $\varphi^{0'}_{f(e)}$ is a total extension function if and only if W_e is infinite. Therefore, W_e is infinite if and only if $f(e)\in E$, so $0''\leq_T E$. We can now see that $0''\leq_T G\oplus 0'$, so G must be high. □

This result makes the following corollary of Theorem 2.6 nonvacuous.

Corollary 2.9. *Let G be a high weakly 2-generic real. Then there is a recursively random (and thus Schnorr random) real A such that $A\leq_{wtt} G$.*

3. Hyperimmunity and Schnorr randomness

In this section and the next, we will make use of the following fact.

Fact 3.1. Let $\langle D_n\rangle_{n\in\omega}$ be a list of the canonical finite sets, and suppose that f is a recursive function from ω to ω and B is hyperimmune. Then there are infinitely many $n\in\omega$ such that $B\cap\{0,\dots,f(n)\}=D_n\cap\{0,\dots,f(n)\}$.

Theorem 3.2. *Let A and B be reals. If A is not high, B is hyperimmune, and $A \leq_{wtt} B$, then A cannot be Schnorr random.*

Proof. Kjos-Hanssen, Merkle, and Stephan showed that a real is complex if and only if it is not wtt-reducible to a hyperimmune-free real in [8]. Since every Martin-Löf random real is complex [8], A cannot be Martin-Löf random. Furthermore, since the Martin-Löf random reals and Schnorr random reals coincide in the non-high degrees, A cannot be Schnorr random, either. □

The following corollary is immediate.

Corollary 3.3. *Let A and B be reals. If B is hyperimmune and not high and $A \leq_{wtt} B$, then A cannot be Schnorr random.*

We may compare this corollary to Theorems 2.1 and 2.4. In these theorems, we required that our real be not only nonhigh and hyperimmune but at least 1-generic. However, we were able to loosen the requirements on the reducibility and consider Turing reducibility instead of simply weak truth-table reducibility.

We can also use Fact 3.1 to prove the following theorem. Since every 1-generic real is hyperimmune, this is a stronger theorem than Theorem 2.5.

Theorem 3.4. *If B is hyperimmune and $A \leq_{tt} B$, then A cannot be recursively random (and thus it cannot be Schnorr random).*

Proof. Suppose that Ψ is a tt-functional that computes A from B, and partition the integers into consecutive intervals $\langle I_n \rangle_{n \in \omega}$ such that the length of I_n is $2n+1$. We will produce a recursive martingale d that witnesses the non-Schnorr randomness of A.

To build this martingale, we first define $f : \omega \to \omega$ to be the function that maps each integer n to the use of the computation of the elements of I_n via Ψ. Since Ψ is a tt-reduction, we may assume that f is recursive.

This martingale d will have an initial capital of 2 and allot $\frac{1}{2^{n+1}}$ to the interval I_n for each n. To determine the behavior of d on the interval I_n, we consider $\Psi^{D_n \upharpoonright f(n)}$. On each bit m of I_n, we bet everything we have remaining from our initial capital of $\frac{1}{2^{n+1}}$ for the interval and anything we have earned on I_n by this point on the value $\Psi^{D_n \upharpoonright f(n)}(m)$. Since $f(n)$ is an upper bound on the use for Ψ, this value will always exist and can be found recursively.

By Fact 3.1, there will be infinitely many n such that $B \upharpoonright f(n) = D_n \upharpoonright f(n)$. Therefore, there will be infinitely many n such that d will bet correctly on every bit of A in the interval I_n. If I_n is such an interval, d will earn $\frac{1}{2^{n+1}} \cdot 2^{2n+1} = 2^n$ on I_n. If I_n is not such an interval, d will lose $\frac{1}{2^{n+1}}$ on I_n. Since the total possible loss is bounded above by 1 and 2^n will be gained for infinitely many n, the recursive martingale d will succeed on A, so A cannot be recursively random. □

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