RELATIVIZATION IN RANDOMNESS

JOHANNA N.Y. FRANKLIN

Abstract. We discuss the different contexts in which relativization appears in randomness and the effect that the relativization chosen has on the results we can obtain. We study several characterizations of the K-trivials in terms of topics ranging from cuppability to density, and we consider a uniform relativization for randomness that gives us more natural results for computable, Schnorr, and Kurtz randomness than the classical relativization does. We then evaluate the relativizations we have considered and suggest some avenues for further work.

1. Introduction

Relativization is one of the cornerstones of the classical theory of algorithmic randomness: it allows us to think about randomness not only as a property that individual sequences possess but to consider the ways that we can use knowledge of one sequence to derandomize another. We can then address questions such as the following:

- Does our sequence A derandomize any random sequences when it is used as an oracle? That is, is it low for randomness?
- Does A have at least as much derandomizing power as B? That is, is A reducible to B if we define a reducibility based on sequences' strengths as oracles?
- Under what conditions will $A \oplus B$ be random? Is it possible for this to happen if A and B can derandomize each other?

In this paper, we will explore relativization in the study of algorithmic randomness through an examination of lowness for randomness and van Lambalgen's Theorem. We introduce the underlying definitions necessary to understand these topics in this section. The next section, Section 2, contains a synopsis of classical results in these areas. We demonstrate that while the classical relativization gives us the expected and desired results for Martin-Löf randomness, it does not do so for computable randomness, Schnorr randomness, and Kurtz randomness. At this point, we begin our discussion of results that arise from the classical relativization for Martin-Löf randomness by investigating one of the most important classes that has arisen from the study of randomness, the K-trivial sequences, in Section 3; this class has been found to have many characterizations that, at first glance, have very little to do with lowness. Then, in Section 4, we go on to discuss an alternate relativization that reduces to the classical relativization for Martin-Löf randomness, thus preserving our original results for that randomness notion and also giving us more intuitively correct results for the others. Finally, in Section 5, we summarize our discussion and present some future directions for work in this area.

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1.1. **Notation.** Our notation will generally follow [19] and [68]. We use lowercase Roman letters such as n and m to represent natural numbers, lowercase Greek letters such as σ and τ to represent finite binary strings, and uppercase Roman letters such as A and B to represent subsets of the natural numbers ω. Often, we will associate a subset of the natural numbers A with its characteristic function χ_A, which allows us to represent A as the infinite binary sequence $\chi_A(0)\chi_A(1)\chi_A(2)\dots$. This lets us discuss subsets of ω and infinite binary sequences interchangeably. If we wish to consider the initial segment of length n of a binary sequence A, we will write $A\restriction n$, and if we want to discuss the length of a finite binary string σ, we will write $|\sigma|$. When written without qualification, a "sequence" may be taken to be infinite and a "string" may be taken to be finite.

We will generally work in the Cantor space, which is the probability space of infinite binary sequences. We say that the basic open set defined by a finite binary string σ is $[\sigma] = \{X \in 2^\omega \mid \sigma \prec X\}$: the set of all infinite binary sequences extending σ. When we discuss measure in the Cantor space, we will always use Lebesgue measure, or the standard "coin-flip" measure, and symbolize this with μ: the Lebesgue measure of a basic open set $[\sigma]$ is $2^{-|\sigma|}$, or the fraction of infinite binary sequences that begin with σ. Occasionally, we may treat a sequence as an element of the interval $[0, 1]$, in which case we may refer to it as a *real*.

We will also adopt notation for the classes of random sequences that we will consider most often in this paper: the classes of Martin-Löf, computable, Schnorr, Kurtz, and weakly 2-random sequences will be denoted by ML, Comp, $\mathsf{Schnorr}$, Kurtz, and $\mathsf{W2R}$, respectively.

Throughout this paper, the name "Miller" will refer to J. Miller.

1.2. **Lowness.** The study of lowness is centered in the following question: Which sequences carry so little useful information that they provide no benefit when used as oracles? This is one of the simplest contexts in which we can discuss relativization. We begin with a relativizable class $\mathcal{C}$ and say that a set is *low for* $\mathcal{C}$ if the class generated using the set as an oracle is precisely the same as the class generated using no oracle at all, in other words, if $\mathcal{C}^A = \mathcal{C}$, and we denote the sets that are low for $\mathcal{C}$ by $\mathrm{Low}(\mathcal{C})$. The term "lowness" was first introduced in 1972 [67] and applied to the class of Turing functionals. In other words, a set A is low if the Turing jump of A is equivalent to the Turing jump of $\emptyset$. Over the past four and a half decades, this notion has been generalized to other areas such as computational learning theory [66] and computable structure theory [26, 30], but it has been most thoroughly developed in the context of algorithmic randomness, as we discuss here. For instance, randomness is the only context in which lowness for pairs of relativizable classes has ever been discussed; we present this concept immediately because so many proofs, even of characterizations of $\mathrm{Low}(\mathcal{C})$ for certain $\mathcal{C}$s, require it.

The concept of lowness for pairs, first introduced in [42], is best illustrated by an example. We already know that every Martin-Löf random sequence is Schnorr random, that is, that $\mathsf{ML} \subset \mathsf{Schnorr}$. If a sequence A is low for Schnorr randomness, we know that $\mathsf{Schnorr}^A = \mathsf{Schnorr}$ and thus that $\mathsf{ML} \subset \mathsf{Schnorr}^A$. However, it may be possible to find a sequence B that is not low for Schnorr such that $\mathsf{ML} \subseteq \mathsf{Schnorr}^B$. Any sequence B for which $\mathsf{ML} \subseteq \mathsf{Schnorr}^B$ is said to be in $\mathrm{Low}(\mathsf{ML}, \mathsf{Schnorr})$.

We generalize this example to the more general concept of lowness for pairs: we say that A is in $\mathrm{Low}(\mathcal{C}, \mathcal{D})$ if $\mathcal{D}^A$ still contains $\mathcal{C}$. We note that $\mathrm{Low}(\mathcal{C}, \mathcal{C})$ is the same as $\mathrm{Low}(\mathcal{C})$ and that it makes no sense to consider $\mathrm{Low}(\mathcal{C}, \mathcal{D})$ unless $\mathcal{C} \subseteq \mathcal{D}$: if $\mathcal{C}$ is not a subset of $\mathcal{D}$, then it can certainly never be a subset of $\mathcal{D}^A$. We can also see that, given a chain of classes $\mathcal{C} \subseteq \widehat{\mathcal{C}} \subseteq \widehat{\mathcal{D}} \subseteq \mathcal{D}$, we will have $\mathrm{Low}(\widehat{\mathcal{C}}, \widehat{\mathcal{D}}) \subseteq \mathrm{Low}(\mathcal{C}, \mathcal{D})$.

Which sequences, then, can belong to Low($\mathcal{C}$) when $\mathcal{C}$ is a class of random sequences? We begin by observing that for any A, $\mathcal{C}^A$ will necessarily be a subclass of $\mathcal{C}$. For instance, if we take A to be a Martin-Löf random sequence, it cannot be low for the class ML.

It will be useful to define lowness in terms of two of the customary frameworks for randomness, beginning with the test framework.

Definition 1.1. Let A be a sequence. A is *low for Martin-Löf (Schnorr) tests* if for every Martin-Löf test relative to A $\langle V_i^A \rangle$, there is an unrelativized Martin-Löf (Schnorr) test $\langle T_i \rangle$ such that $\cap_i V_i^A \subseteq \cap_i T_i$.

The other framework in which lowness is generally defined is that of Kolmogorov complexity.

Definition 1.2. Let A be a sequence.
(1) A is *low for K* if there is some c such that for all σ, $K(\sigma) \leq K^A(\sigma) + c$ (unpublished, from Muchnik [19]).
(2) A is *low for computable measure machines* if for every A-computable measure machine M, there is a computable measure machine M' and a constant c such that for all σ, $K_{M'}(\sigma) \leq K_M(\sigma) + c$ [13].

While lowness is one way of characterizing a sequence as information poor, there are ways other than lowness in which this can happen in randomness. One way is for the sequence to have initial-segment complexity no higher than that of 0^ω:

Definition 1.3. Let A be a sequence.
(1) A is *K-trivial* if there is some c such that for all n, $K(A\restriction n) \leq K(0^n) + c$ [8].
(2) A is *Schnorr trivial* if for every computable measure machine M, there is a computable measure machine M' and a constant c such that for all n, $K_{M'}(A\restriction n) \leq K_M(0^n) + c$ [14].

We note that none of these definitions apply to computable randomness or Kurtz randomness. Neither of them is known to have a natural definition in terms of tests or Kolmogorov complexity, so we will present appropriate definitions for them as necessary later.

Finally, we present the least-studied way in which a sequence can be information poor.

Definition 1.4. [45] Let A be a sequence. A is a *base* for Martin-Löf randomness if there is a sequence $B \geq_T A$ such that B is Martin-Löf random relative to A.

1.3. **van Lambalgen's Theorem.** We can now consider the role that relativization plays in preserving randomness when two random sequences are combined in a computable way. We can see easily that their randomness alone is not enough to guarantee the randomness of their combination. For instance, $A \oplus A$ will never be random, and $A \oplus B$ may not be random even if A and B are distinct. For instance, suppose that $A \leq_T B$. Since B can compute A, it may be possible to predict enough of the bits of A in $A \oplus B$ from the bits of B that have already appeared to make $A \oplus B$ nonrandom.

This question was answered by van Lambalgen for Martin-Löf randomness in 1990 [72]. Intuitively, the answer to this question for Martin-Löf randomness is precisely what it should be. For $A \oplus B$ to be Martin-Löf random, not only do both A and B have to be Martin-Löf random, they must be random relative to each other. We state this result formally here:

van Lambalgen's Theorem. [72] The following statements are equivalent.

(1) A is Martin-Löf random relative to B and B is Martin-Löf random relative to A.
(2) $A \oplus B$ is Martin-Löf random.

In Section 2, we will discuss the extent to which these implications hold for other types of randomness under the standard relativization.

2. Results with the standard relativization

Here, we present a brief overview of the "classical" results concerning lowness for randomness and van Lambalgen's Theorem. As we go through these results, we should note (in the case of lowness) whether the classes under consideration are related to each other as we would expect and (in the case of van Lambalgen's Theorem) whether both directions, if either, of its equivalence hold. In later sections, we will consider whether alternate relativizations result in more plausible results.

2.1. **Lowness.** The results mentioned here can be found in Chapters 11 and 12 of [19] and are summarized in [22]. Some of the short proofs may be sketched; the longer ones will simply have "ingredient lists."

The question is now which Turing degrees are low for randomness and how the various concepts discussed in Subsection 1.2 are related to each other. If all of these concepts coincide for a given relativization, it suggests that the chosen relativization is a sensible one. It is straightforward to see that lowness for Martin-Löf tests is equivalent to lowness for Martin-Löf randomness since there is a universal Martin-Löf test. Being low for K, being K-trivial, and being a base for Martin-Löf randomness are also equivalent to lowness for Martin-Löf randomness, but more work is required to show these facts. Nies proved the equivalence of lowness for K and lowness for Martin-Löf randomness in [59]. This paper also contains a proof by Nies and Hirschfeldt that K-triviality and lowness for K are equivalent. We will begin by discussing these two results; afterwards, we will discuss Hirschfeldt, Nies, and Stephan's proof of the equivalence of K-triviality and being a base for Martin-Löf randomness from [34].

It is relatively straightforward to see that lowness for K implies lowness for Martin-Löf randomness: since ML can be defined in terms of the prefix-free Kolmogorov complexity K, ML^A can be defined in terms of K^A, so every sequence that is low for K is low for Martin-Löf randomness. The proof that lowness for Martin-Löf randomness is equivalent to lowness for K is most easily done by considering two reducibilities: $\leq_{LK}$ and $\leq_{LR}$, introduced in [59].

Definition 2.1. Let A and B be sequences. We say that $A \leq_{LK} B$ if for all σ, there is some c such that $K^B(\sigma) \leq K^A(\sigma) + c$ and that $A \leq_{LR} B$ if $\mathsf{ML}^B \subseteq \mathsf{ML}^A$.

Both of these reducibilities let us compare the derandomization abilities of two sequences: in each case, A is reducible to B if B can derandomize at least as much as A in a certain context. Since Martin-Löf randomness can be defined in terms of Kolmogorov complexity, we can see that if $A \leq_{LK} B$, then $A \leq_{LR} B$. Kjos-Hanssen, Miller, and Solomon showed that LR-reducibility implies LK-reducibility, so we have the reverse direction as well [41], and these reducibilities are actually identical.

We now turn our attention to K-triviality. Showing that lowness for K implies K-triviality is relatively straightforward [59]: given a constant c_0 witnessing A's lowness for K and a universal prefix-free machine U, we simply build a universal prefix-free machine M such that for every X and σ, $M^X(\sigma) = X \upharpoonright |U(\sigma)|$ whenever this is defined. If a sequence A is low for K, this allows us to first pass from information about the complexity that the unrelativized machine U gives an initial segment of A to information about the complexity that the A-relativized M gives it to information about the complexity of a string of 0s of the same length for only the cost of a constant.

Showing that K-triviality implies lowness for K, though, is far more difficult and makes use of the "golden run" method.

Theorem 2.2. [59] *Every sequence that is K-trivial is low for K.*

The proof of this result is rather complicated, so we will simply give an idea of the ingredients of the proof and how they fit together.

We begin with a computable approximation to our K-trivial sequence A. Our primary tools are two Kraft-Chaitin sets, L and W. The set L will allow us to use A's K-triviality in our proof, and W will witness A's lowness for K.

The construction involves a tree of "runs" of procedures. At each branching point, we try to ensure that W generates a prefix-free machine that will witness A's lowness for K. The complications arise when A's approximation changes. We cannot simply compensate for this change every time; if we do, L will be too large. Therefore, we refuse to allow an axiom to enter L until it is "approved" at a certain level of the tree.

With this result, we have shown that lowness for Martin-Löf randomness, lowness for Martin-Löf tests, lowness for K, and K-triviality are all equivalent. We now turn our attention to the last property: being a base. Kučera observed that the Kučera-Gács Theorem [31, 44] can be applied to show that if A is low for Martin-Löf randomness, then it is a base for Martin-Löf randomness. Hirschfeldt, Nies, and Stephan completed the proof of this equivalence:

Theorem 2.3. [34] *If a sequence is a base for Martin-Löf randomness, then it is K-trivial.*

Here, we fix a sequence $B \geq_T A$ that is A-Martin-Löf random and enumerate a Kraft-Chaitin set L_n for each $n \in \omega$. We will define uniformly Σ^0_1 classes that are pairwise disjoint and have small measure; these will allow us to build an A-Martin-Löf tests. The L_n are then defined based on these Σ^0_1 classes, and one of these L_ns will witness A's K-triviality.

Now we have seen that all the mentioned characterizations of being far from Martin-Löf random coincide using the standard relativization, and as is usually done, we will call this class of sequences $\mathcal{K}$. We note without ceremony that all elements of $\mathcal{K}$ are Δ^0_2 and, in fact, superlow [59].

We now turn our attention to classes of the form $\text{Low}(-, \mathsf{Comp})$. Nies proved the following results in [59]. As hinted at in the beginning of the section, the characterization of $\text{Low}(\mathsf{ML}, \mathsf{Comp})$ is necessary for his proof of the characterization of $\text{Low}(\mathsf{Comp})$.

Theorem 2.4. [59] *A sequence A is in $\text{Low}(\mathsf{ML}, \mathsf{Comp})$ if and only if A is low for K.*

We note that since $\mathsf{ML} \subseteq \mathsf{ML} \subseteq \mathsf{ML} \subseteq \mathsf{Comp}$, we have that $\text{Low}(\mathsf{ML}) = \text{Low}(\mathsf{ML}, \mathsf{ML}) \subseteq \text{Low}(\mathsf{ML}, \mathsf{Comp})$ immediately. Therefore, we simply need to show that any element of $\text{Low}(\mathsf{ML}, \mathsf{Comp})$ is low for K.

This proof uses the martingale characterization of computable randomness: we construct a martingale functional L such that A is low for K if L^A only succeeds on sequences that are not Martin-Löf random.[1] Rather than build L directly, we set L to be the sum of the martingale functionals L_n, which we actually construct.

We now consider $\text{Low}(\mathsf{Comp})$. Since $\mathsf{ML} \subseteq \mathsf{Comp} \subseteq \mathsf{Comp} \subseteq \mathsf{Comp}$, $\text{Low}(\mathsf{Comp}) = \text{Low}(\mathsf{Comp}, \mathsf{Comp}) \subseteq \text{Low}(\mathsf{ML}, \mathsf{Comp})$. A result of Bedregal and Nies states that every element

[1] A martingale functional is a Turing functional that is a martingale for every oracle.

of Low(Comp) is hyperimmune free [4].[2] Since every sequence that is low for K is Δ^0_2, every element of Low(Comp) must be Δ^0_2, and since the only hyperimmune-free Δ^0_2 sequences are the computable sequences, Low(Comp) consists of precisely the computable sequences.

Although there is no analogue to K-triviality or lowness for K for computable randomness, the bases for computable randomness have been partially characterized with the following two results. Recall that a sequence is DNC if it computes a total function f such that $f(e)$ is never equal to $\Phi_e(e)$ for any e and that a Turing degree is PA if it can compute a complete extension of Peano Arithmetic (see Sections 2.22 and 2.21, respectively, in [19]).

Theorem 2.5. [34] *Every non-DNC Δ^0_2 sequence is a base for computable randomness, but no sequence in a PA degree is.*

Corollary 2.6. [34] *An n-c.e. sequence is a base for computable randomness if and only if it is Turing incomplete.*[3]

To show that a non-DNC Δ^0_2 sequence is a base for computable randomness, we take such a sequence A and construct a generic sequence G such that $(A \oplus G)' \equiv_T A''$. This sequence is used as an oracle to build a $Z \geq_T A$ that is A-computably random. The other half of this theorem is proven by noting that a sequence A with PA degree will compute a sequence B with PA degree such that A is PA relative to B. We then show that if there is a sequence computing A that is A-computably random, we can conclude that B is both K-trivial and not K-trivial and derive a contradiction.

The proof of the corollary uses a theorem in [37] that states that a Turing incomplete n-c.e. sequence is not DNC and is thus a base for computable randomness. Furthermore, such a sequence that is Turing complete will necessarily have PA degree and thus cannot be a base.

While we do not have as many ways to talk about the information poverty of a sequence in the context of computable randomness as we do in that of Martin-Löf randomness, we can still see that the classes of information-poor sequences we do have for computable randomness do not align neatly: Low(Comp) is a proper subset of the bases for computable randomness.

We now turn our attention to lowness for Schnorr randomness. We begin by observing that lowness for Schnorr tests clearly implies lowness for Schnorr randomness; however, the converse is not obvious since there is no universal Schnorr test. In order to characterize the Low($-$, Schnorr) classes, we need a new concept: traceability.

Definition 2.7. [74] A sequence A is *c.e. traceable* if there is an order function[4] p such that for all $f \leq_T A$, there is a computable function r such that the following conditions hold for all n:

- $f(n) \in W_{r(n)}$ and
- $|W_{r(n)}| \leq p(n)$.

A sequence is *computably traceable* if we can replace $W_{r(n)}$, the $r(n)^{th}$ canonical c.e. set, with $D_{r(n)}$, the $r(n)^{th}$ canonical finite set, in the above definition.

[2]A sequence is *hyperimmune free* if every function it computes is dominated by a computable function.

[3]A sequence A is n-c.e. if it has a computable approximation $\langle A_s \rangle$ such that there are no more than n values of s at which $A_s(k)$ and $A_{s+1}(k)$ differ for each k. A sequence A is ω-c.e. if it has a computable approximation $\langle A_s \rangle$ such that for some computable function g, there are no more than $g(k)$ values of s at which $A_s(k)$ and $A_{s+1}(k)$ differ for each k.

[4]An *order function* is a computable function that is increasing and unbounded.

We note that all computably traceable sequences are hyperimmune free and that we can think of a computably traceable sequence as uniformly hyperimmune free.

This notion allowed Terwijn and Zambella to characterize the sequences that are low for Schnorr tests.

Theorem 2.8. [71] *A sequence is computably traceable if and only if it is low for Schnorr tests.*

If we assume that a sequence A is computably traceable, we can take a computable trace for a Schnorr test relative to A and use this trace to define an unrelativized Schnorr test that covers it. If we assume that A is low for Schnorr tests, we take an arbitrary function computable from A and code it into a Schnorr test relative to A using clopen sets; we then show that we can build a computable trace for this function using a single element of an unrelativized Schnorr test that covers our original A-Schnorr test.

Later, Kjos-Hanssen, Nies, and Stephan showed that the computably traceable sequences were precisely those that were low for Schnorr randomness. They began by characterizing Low($\mathsf{ML}, \mathsf{Schnorr}$) using c.e. traceability.

Theorem 2.9. [42] *A sequence is c.e. traceable if and only if it is in Low*($\mathsf{ML}, \mathsf{Schnorr}$)*.*

To prove this, we assume that a sequence A is c.e. traceable, take a Schnorr test relative to A, and use a trace for A to approximate the components of this Schnorr test in a computable way. This approximations allow us to construct a Martin-Löf test whose intersection contains the intersection of the relativized Schnorr test.

Conversely, if A is in Low($\mathsf{ML}, \mathsf{Schnorr}$), we need to construct a c.e. trace for an arbitrary $f \leq_T A$. We code this f into a Schnorr test relative to A and recognize that the intersection of this A-Schnorr test is contained in each element of the universal Martin-Löf test. We can use one element of this unrelativized universal Martin-Löf test to build a c.e. trace for f. Our method for encoding the initial segments of f into the A-Schnorr test ensures that we don't include too many options for each value $f(n)$ while ensuring at the same time that we include the correct value.

Finally, we use this result to find the other classes Low($-, \mathsf{Schnorr}$).

Theorem 2.10. [42] *Let A be a sequence. The following are equivalent:*

(1) *A is computably traceable.*
(2) *A is in Low*($\mathsf{Schnorr}$)*.*
(3) *A is in Low*($\mathsf{Comp}, \mathsf{Schnorr}$)*.*

Terwijn and Zambella had already shown that (1) implies (2), and it can be seen from the relationship between Comp and $\mathsf{Schnorr}$ that (2) implies (3). Therefore, the only thing to do is to show that (3) implies (1). This follows from the same paper of Bedregal and Nies mentioned earlier [4]: they showed that every sequence in Low($\mathsf{Comp}, \mathsf{Schnorr}$) must be hyperimmune free, and since every hyperimmune-free c.e. traceable sequence is actually computably traceable, we are done.

We now consider lowness for computable measure machines. Downey, Greenberg, Mihailović, and Nies showed that this is also equivalent to computable traceability:

Theorem 2.11. [13] *A sequence A is low for computable measure machines if and only if it is computably traceable.*

Since lowness for computable measure machines implies lowness for Schnorr randomness, it is easy to prove the forward direction. To prove the other direction, we assume that A is a computably traceable sequence and choose an A-computable measure machine M^A; we then use the computable traceability of A to "trace" M^A and construct a sequence of small finite sets. We then build a Kraft-Chaitin set from this sequence that allows us to construct a computable measure machine witnessing A's lowness for computable measure machines.

Bases for Schnorr randomness and Schnorr triviality have also been the subject of study. At first glance, the Schnorr trivial sequences appear to be very different from the sequences that are low for Schnorr randomness. Downey, Griffiths, and LaForte proved that there is a Turing complete Schnorr trivial sequence [14], and Franklin proved that, in fact, every high Turing degree contains a Schnorr trivial sequence [21]. The Schnorr trivial Turing degrees do not form an ideal as the K-trivial Turing degrees do since they are not closed downward [59]. However, Franklin showed that the following relationship between Schnorr triviality and lowness for Schnorr randomness holds:

Theorem 2.12. [20] *A sequence is low for Schnorr randomness if and only if it is Schnorr trivial and hyperimmune free.*

To see this, we begin by assuming that A is low for Schnorr randomness and fixing an arbitrary computable measure machine M. We use an enumeration of this machine and the computable traceability of A to construct another computable measure machine M' witnessing the Schnorr triviality of A with respect to M. To prove the other direction, we assume that A is hyperimmune free but not Schnorr low. We then take a function $g \leq_T A$ that witnesses that A is not computably traceable and, since A is hyperimmune free, observe that the use function for this g relative to all possible A is a computable function. Now, for every computable measure machine M, we can define a computable trace that contains all strings of a certain length with low complexity. Since A is not Schnorr low, there is an n for which $g(n)$ is not in this trace, and thus A will not be Schnorr trivial with respect to this M.

We conclude our discussion of lowness for Schnorr randomness with a discussion of bases. Franklin, Stephan, and Yu showed that the bases for Schnorr randomness are precisely the sequences that cannot compute $\mathbf{0}'$ by showing that if $A \not\geq_T \mathbf{0}'$, then there is a set above A that is A-Schnorr random but not computably random [29]; this is clearly a different class from those that are low for Schnorr randomness.

Once again, we can see that these classes of information-poor sequences are not the same in the context of Schnorr randomness. Low($\mathsf{Schnorr}$) is a proper subset of the Schnorr trivial sequences, and the Schnorr trivial sequences and the bases for Schnorr randomness are incomparable as classes.

We take a moment to discuss another approach to lowness that will be adapted later with an alternate relativization: Bienvenu and Miller's open covers [7]. The underlying concept is that of a *bounded* set: an open set is bounded if its measure is smaller than 1. We begin by presenting a result of Kučera that uses this concept to characterize Martin-Löf randomness; we note that the *tail* of a sequence is a sequence obtained by removing some initial segment, and if U is a set of strings, then U^ω is the set of sequences that can be created by concatenating countably many elements of U.

Theorem 2.13. [44] *The following are equivalent for a sequence A:*

(1) *A is not Martin-Löf random.*
(2) *There is a bounded c.e. open set $\mathcal{U}$ such that all tails of A are contained in $\mathcal{U}$.*
(3) *A is contained in U^ω for some bounded c.e. prefix-free set U.*

Bienvenu and Miller proved similar characterizations for computable randomness and Schnorr randomness in [7]. Their version of this result for Schnorr randomness replaces "bounded c.e. open set" with "bounded Schnorr open set," where a Schnorr open set is simply a c.e. open set with computable measure. In the case of computable randomness, this is replaced by "winning open set," where a winning open set is one generated by the minimal strings σ such that for some martingale given by a computable function $d : 2^{<\omega} \to \mathbb{Q}^{\geq 0}$ and some rational $q > 1$, $d(\sigma) \geq q$.

This type of characterization of randomness notions in terms of a single open set rather than a standard test or martingale allows us to define lowness in terms of these open sets. For instance, A is in Low($\mathsf{Schnorr}$) exactly when every bounded A-computable open set can be covered by a bounded computable open set, and A is in Low($\mathsf{ML}, \mathsf{Schnorr}$) exactly when every bounded Schnorr open set can be covered by a bounded c.e. open set.

The last type of randomness which we will consider in this context is Kurtz randomness. We begin with the characterization of Low(Kurtz).

In [18], Downey, Griffiths, and Reid showed that every sequence that is low for Schnorr randomness is also low for Kurtz tests and that all sequences that are low for Kurtz tests are hyperimmune free using methods very much like Terwijn and Zambella's [71]. Later, Stephan and Yu proved that any sequence that is hyperimmune free and not DNC is low for Kurtz randomness [70], and then Greenberg and Miller proved that any sequence that is DNC cannot be low for Kurtz randomness [32]. While Miller later gave a shorter proof of the latter result, we will present the Greenberg/Miller proof here because the discussion will be useful in Section 4.

Theorem 2.14. [70, 32] *A sequence is low for Kurtz randomness if and only if it is hyperimmune free and not DNC.*

If we suppose that A is hyperimmune free and not DNC and that U^A is a Σ^A_1 class of measure 1, we can use the density of U^A to identify sets of long strings whose neighborhoods are entirely contained in U^A and yet have large measure computably in A. These allow us to build a Σ^0_1 class of measure 1 that is contained by U^A.

The other direction involves the use of svelte trees; rather than define them formally, we will just say that a svelte tree is a finite subtree of $\omega^{<\omega}$ that can be covered by so few basic clopen sets that none of its paths can be made to be DNC. Then, to show that a sequence that is low for Kurtz tests is not DNC, we show that for any DNC function f, there is a Π^f_1 null class that is not contained in any Π^0_1 null class. The result for lowness for Kurtz randomness follows quickly.

We note without ceremony that lowness for pairs of randomness notions involving Kurtz randomness have also been characterized in [32]; this paper contains a result by Kjos-Hanssen that is necessary for the proof of the theorem immediately below.

Theorem 2.15. [32] *Low*($\mathsf{ML}, \mathsf{Kurtz}$) *consists of precisely the sequences that are not DNC.*

Theorem 2.16. [32] *The following are equivalent for a sequence A:*

1. *A is not high or DNC.*
2. *A is in Low*($\mathsf{Comp}, \mathsf{Kurtz}$)*.*
3. *A is in Low*($\mathsf{Schnorr}, \mathsf{Kurtz}$)

A summary of these lowness classes can be found in Table 1. Lowness for other randomness notions such as weak 2-randomness and difference randomness have been studied as well [16, 25, 41], but we will not consider these in this survey since they do not cast light on the role of relativization

	ML	Comp	Schnorr	Kurtz
ML	K-trivial	K-trivial	c.e. traceable	not DNC
Comp		computable	computably traceable	not high or DNC
Schnorr			computably traceable	not high or DNC
Kurtz				hyperimmune free & not DNC

TABLE 1. Characteristics of lowness classes: standard relativization

in the same way that a comparison of lowness for Martin-Löf randomness and Schnorr randomness, in particular, does.

2.2. **van Lambalgen's Theorem.** We recall van Lambalgen's Theorem for Martin-Löf randomness:

van Lambalgen's Theorem. [72] The following statements are equivalent.

(1) A is Martin-Löf random relative to B and B is Martin-Löf random relative to A.
(2) $A \oplus B$ is Martin-Löf random.

The claim that (1) implies (2) is known as the "hard" direction, and the claim that (2) implies (1) is known as the "easy" direction.

To prove the "easy" direction, we simply assume that B is not Martin-Löf random relative to A and that thus $B \in \cap_i U_i^A$ and use this to produce a Solovay test[5] that contains $A \oplus B$. This contradicts the assumption that $A \oplus B$ is Martin-Löf random.

One of the standard proofs of the "hard direction" is by Nies [15]. We assume for a contradiction that $A \oplus B$ is not Martin-Löf random and, from a Martin-Löf test containing $A \oplus B$, produce a sequence of sets $\langle S_i \rangle$. We then construct a Martin-Löf test based on these S_is. If this test contains A, then we have a contradiction because A cannot even be Martin-Löf random unrelativized; if it does not, we can use this information about A to construct an A-Martin-Löf test that contains B.

We now note an interesting porism of van Lambalgen's Theorem:

Porism 2.17. *A is Martin-Löf random relative to B if and only if B is Martin-Löf random relative to A.*

In short, if two sequences are Martin-Löf random and the first is random relative to the second, the second must also be random relative to the first.

However, we can also consider van Lambalgen's Theorem in the contexts of computable and Schnorr randomness. For a long time, it was assumed that the "hard" direction of van Lambalgen's Theorem was also true for computable and Schnorr randomness "with essentially the same proof" (p. 276 of [19]). However, no formal proof of either statement was given until 2011, when Franklin and Stephan proved that the "hard" direction holds for Schnorr randomness.

Theorem 2.18. [28] *If A is Schnorr random and B is Schnorr random relative to A, then $A \oplus B$ must be Schnorr random.*

Their proof, like Nies's for the Martin-Löf case, involves supposing that A is Schnorr random, B is A-Schnorr random, and $A \oplus B$ is not Schnorr random for a contradiction. However, they use

[5] A Solovay test gives us another measure-theoretic characterization of Martin-Löf randomness [69]. For the moment, simply note that anything contained in a Solovay test is not Martin-Löf random.

martingales instead of the test characterization that Nies used. They use a computable martingale that succeeds on $A \oplus B$ in the sense of Schnorr to construct a set S of lengths of strings on which this martingale has values above a certain level. If there are infinitely many lengths in S with a certain technical property, then A can be shown to be not Schnorr random; if there are finitely many, then B can be shown to be not A-Schnorr random.

Miyabe and Rute have an alternate proof of this result in [58] that uses integral tests.[6] They begin with a Schnorr integral test witnessing that $A \oplus B$ is not Schnorr random and use this to construct a Schnorr integral test relative to A witnessing that B is not Schnorr random relative to A.

The corresponding result for computable randomness was not obtained until a few years later, when Bauwens proved that the "hard" direction does not hold in this situation: he constructed a sequence $A \oplus B$ that was not computably random itself, though A was computably random and B was computably random relative to A [3]. In his proof, the sequence A is chosen arbitrarily, and an enumeration of all rational partial martingales with oracle A is determined. The sequence B encodes information about the totality of each of these martingales and their computation times in strings that are assigned low values by the martingales on this list. This ensures that B will be computably random relative to A and encodes enough information about $A \oplus B$ that it can be predicted sufficiently well by a computable martingale and will not be computably random itself.

Bauwens further notes that with a slightly stronger hypothesis, the "hard" direction is in fact true for computable randomness: If A and B are computably random relative to each other, then $A \oplus B$ is computably random [3]. The proof proceeds by computably decomposing a martingale into two different martingales, one that bets only on the even bits and one that bets only on the odd bits, and then transforming these into conditional martingales.

On the other hand, it has long been known that the "easy" direction of van Lambalgen's Theorem does not hold for Schnorr or computable randomness; namely, that there is a Schnorr (computably) random sequence $A \oplus B$ such that at least one of A and B is not Schnorr (computably) random relative to the other. While the first result to this effect appears in [50], the following argument appearing in [61], due to Kjos-Hanssen, is simpler.

A *high* Turing degree is one with an element that *wtt*-computes a function that dominates every computable function. The Cooper Jump Inversion Theorem states that a minimal high Turing degree exists [9], and every high Turing degree contains a computably random sequence $A \oplus B$ [62]. Since A and B are Turing computable from $A \oplus B$ and $A \oplus B$ belongs to a minimal degree, A and B must either be in the same Turing degree as $A \oplus B$ or computable. Clearly, they must both be in the same Turing degree as $A \oplus B$. However, this means that they are mutually computable and therefore neither of them is computably random relative to the other. Since every high degree contains a Schnorr random sequence that is not computably random [62], this argument holds for Schnorr randomness as well.

These are not the only results that have been obtained in this area. Yu proved that this direction of van Lambalgen's Theorem fails for computably random sequences bounded below $\mathbf{0}'$ by a c.e. set [73], and Franklin and Stephan proved that every high degree contains a computably random sequence for which van Lambalgen's Theorem fails for both computable and Schnorr randomness [28].

[6] An *integral test* is a lower semicomputable function on 2^ω whose integral is bounded (in the case of an Martin-Löf integral test) or computable (in the case of a Schnorr integral test). This originates in [48] and [55] and will be defined again in Section 4 when a more formal definition is necessary.

Franklin and Stephan's proof in particular sheds light on the reason that a Schnorr or computably random sequence does not necessarily decompose into two mutually Schnorr or computably random sequences. They begin with an arbitrary high degree and a set A in it that *wtt*-computes a function that dominates all computable functions. As a preliminary, they use this dominating function to construct a martingale m that subsumes all computable martingales; that is, if any computable martingale succeeds on a sequence, m will too. They then construct a sequence B that is contained in that high degree on which m does not succeed. This sequence will therefore be computably random.

To ensure that van Lambalgen's Theorem fails for B, they construct it in finite initial segments, alternating two kinds of steps in the construction. In the first kind of step, they extend the current finite segment of B by another that has two properties:

(1) m does not increase very much on this new part of B and
(2) this new part of B codes an initial segment of the set A.

The first condition ensures that B is not computably or Schnorr random itself; the second ensures that $A \leq_T B$. The rest of the construction makes it clear that $B \leq_T A$: a martingale that is computable from A determines B, and all other aspects of the construction are themselves computable.

In the second kind of step, the current initial segment σ of B is extended by a single bit that can be determined from σ. The locations of these bits can be calculated computably, which determines the way the sequence is determined for the join: the locations of these bits form a computable set F. If B is separated into two pieces, one comprised of the bits at locations in F and one comprised of the bits at locations in F^c, we get two infinite binary sequences. We call the sequence formed by the F-bits X and the sequence formed by the F^c-bits Y. Clearly, $X \oplus_F Y = B$. However, $X \leq_T Y$, so X is not computably random relative to Y. In fact, a slight additional argument can be used to show that X is not even Schnorr random relative to Y (this argument is necessary because X can still be Schnorr random relative to Y if X is computable from Y as long as the computation of X from Y is very slow).

Now that we have seen this proof, we can see why this direction of van Lambalgen's Theorem can fail for computable or Schnorr randomness. Suppose that we have a computably random sequence B that can be written as $X \oplus Y$, where X is not computably random relative to Y. This means that there must be a computable martingale m' relative to Y that succeeds on X. How, then, could that martingale not be converted to a computable martingale that would succeed on $X \oplus Y$? There are two possible ways. The first is the simplest: it may not be possible to define such a martingale on $X \oplus Z$ for every sequence Z. If that happens, then the result will not be a computable martingale. The second is inherent in the argument above: perhaps m' would have to see too many bits of Y before betting on a bit of X. In Franklin and Stephan's proof, their choice of F guarantees that there is a computable bound on the amount of Y that must be seen before a given bit of X is determined. If no such bound exists, a computable martingale cannot be found.

3. Further characterizations of Low(ML)

We turn our attention now to $\mathcal{K}$ as a class and further demonstrate its robustness by showing that it has a number of other characterizations involving concepts ranging from lowness notions to analytic notions such as density to classical degree-theoretic notions such as cuppability.

The first characterization we consider involves stabilizing sets. A set A can be said to derandomize a set X if $A \triangle X$ is not random, which led Kuyper and Miller to the following definition:

Definition 3.1. [47] If $\mathcal{C} \subseteq 2^\omega$, then the *stabilizer* of $\mathcal{C}$ is the set

$$\{A \in 2^\omega \mid (\forall X \in \mathcal{C})(A \triangle X \in \mathcal{C})\},$$

and an element of this set is said to be $\mathcal{C}$ stabilizing. Furthermore, if $\mathcal{C}$ and $\mathcal{D}$ are both subsets of 2^ω, a sequence A is said to be $(\mathcal{C}, \mathcal{D})$ stabilizing if whenever $X \in \mathcal{C}$, we have that $A \triangle X \in \mathcal{D}$.

They then used this definition to characterize $\mathcal{K}$:

Theorem 3.2. [47] *A sequence A belongs to $\mathcal{K}$ if and only if it is Martin-Löf stabilizing.*

One direction of the proof is trivial: any set that is low for Martin-Löf randomness must be Martin-Löf stabilizing. The other direction is more complicated. We choose a universal Martin-Löf test $\langle U_i \rangle$ such that every Σ^0_1 class W_e is contained in any U_i such that $\mu(W_e) \leq 2^{-(e+i+2)}$, and we let Q_i be the complement of U_i. We begin by showing that for any stabilizing set B, there is a Π^0_1 class P of positive measure and an $m \in \omega$ such that $A \triangle P \subseteq Q_m$. Our goal is to construct a Π^0_1 class R that contains Q_m and a Kraft-Chaitin set L; the latter will be used to show that B must be K-trivial.

To do this, we assume that we know an index e for R by the recursion theorem and construct R so that its measure is at least $1 - 2^{-(e+m+2)}$; the complement of R will be contained in W_m and thus $Q_m \subseteq R$. It will then suffice to show that whenever $B \triangle P \subseteq R$, L witnesses that B must be K-trivial.

Kuyper and Miller define a fast-growing auxiliary function $f(k, n)$ that defines segments in ω; we remove all strings that are 0 along one of these segments from R if they correspond to a decrease in the approximation of a number's Kolmogorov complexity (we identify the number n with the string 0^n). More precisely, at stage s, if we find some n such that $k = K_s(n) < K_{s-1}(n)$, we remove the elements of R extending strings of length n with a string of zeroes in a certain subinterval of $[f(\langle k, n\rangle - 1), f(\langle k, n\rangle))$. Then we add (k, σ) to L if $|\sigma| = n$ and there is a string τ of the appropriate length such that $\tau \triangle \overline{P_s}$ contains all the strings we have just removed.

To see that this works, we can suppose that A is Martin-Löf stabilizing and thus that $A \triangle P \subseteq R$. If $k = K(n)$, we can take $A \restriction f(\langle n, k\rangle)$ and, after the stage at which we act for $\langle n, k\rangle$, we will have $A \restriction f(\langle n, k\rangle) \triangle P \subseteq R$, and we will have therefore enumerated a request $(k, A \restriction f(\langle n, k\rangle)$ into L. This will guarantee that $K(A \restriction n) \leq K(n) + c$ for the c associated with L.

The authors then go on to describe a modification of this argument that lets them prove the following for the pair $(\mathsf{W2R}, \mathsf{ML})$:

Theorem 3.3. [47] *A sequence A belongs to $\mathcal{K}$ if and only if it is weakly 2-random stabilizing if and only if it is* $(\mathsf{W2R}, \mathsf{ML})$ *stabilizing.*

Now we turn from a notion defined by a way of removing information from a sequence to a long-standing topic in degree theory, cuppability, and a variant of it involving Martin-Löf randomness.

Definition 3.4. A Turing degree $\mathbf{a}$ is *cuppable* to another degree $\mathbf{d}$ if there is a degree $\mathbf{b}$ such that $\mathbf{a} \cup \mathbf{b} = \mathbf{d}$. Furthermore, a sequence A is *weakly ML-cuppable* if there is a Martin-Löf random sequence $Z \not\geq_T 0'$ such that $A \oplus Z \geq_T 0'$. If there is such a $Z <_T 0'$, A is said to be *ML-cuppable.*

In 2004, Kučera asked which Δ^0_2 sets were (weakly) ML-cuppable and if either ML-cuppability or weak ML-cuppability were equivalent to not belonging to $\mathcal{K}$ (Question 4.8 in [52]). It was known that any Δ^0_2 set not in $\mathcal{K}$ is weakly ML-cuppable since $\mathcal{K}$ is the set of bases for Martin-Löf randomness, and Nies showed that a c.e. set that is not ML-cuppable must belong to $\mathcal{K}$ [60]. Kučera's questions have now been answered fully by Day and Miller in the following two theorems:

Theorem 3.5. [10] *If $A <_T 0'$, then A is ML-cuppable if and only if $A \notin \mathcal{K}$.*

Theorem 3.6. [10] *For any sequence A, A is weakly ML-cuppable if and only if $A \notin \mathcal{K}$.*

They begin by proving that no $A \in \mathcal{K}$ is weakly ML-cuppable using a relativization of a result concerning the Turing complete Martin-Löf random sets by Bienvenu, Hölzl, Miller, and Nies in terms of Π^0_1 classes: for a given set A, if X is A-Martin-Löf random and $A \oplus X \geq_T A$, then there is a $\Pi^0_1(A)$ class P that contains X such that X has lower density zero in P [6].[7]

The other primary ingredient is a result on bounded sets of finite strings implicit in Miller, Kjos-Hanssen, and Solomon [41] and explicitly stated by Simpson in [64]. Here, we say that a set S of strings is bounded if $\sum_{\sigma \in S} 2^{-|\sigma|} < \infty$, and the necessary result is something of a covering property: for any $A \in \mathcal{K}$ and W_A a bounded set of strings that is c.e. in A, there is a bounded c.e. set of strings W such that $W_A \subseteq W$.

Now this direction proceeds easily. If $A \in \mathcal{K}$, R is Martin-Löf random, and $A \oplus R \geq_T 0'$, then we must have $A \oplus R \geq_T A'$ and R must be A-Martin-Löf random because every K-trivial is low. We can thus find the $\Pi^0_1(A)$ set P_A required by Bienvenu, Hölzl, Miller, and Nies's result. Now we let W_A be the A-c.e. prefix-free set defining P_A. This set necessarily has bounded weight, so we can use our other result to see that there is a c.e. set $W \supseteq W_A$ with bounded weight. Such a set is a Solovay test, and since R is Martin-Löf random, there are finitely many initial segments of R in W. Since none of them are in W_A, we can remove them from W and preserve $W_A \subseteq W$. We now take P to be the Π^0_1 set defined by W. Since $R \in P$ and $P \subseteq P_A$, R must have lower density zero in P as well as in P_A.

The next step is to show that for any set $A \notin \mathcal{K}$, there is a Martin-Löf random $R \not\geq_T 0'$ such that $A \oplus R \geq_T 0'$. Their proof will give us Theorem 3.5, and a closer analysis of the proof lets us see that if we drop the requirement that A and an auxiliary set D are below $0'$, we have Theorem 3.6.

We begin with a set $A \notin \mathcal{K}$ that is strictly below $0'$ and an auxiliary set D such that $0 <_T D \leq_T 0'$ and construct a low Martin-Löf random R that does not compute D and is a cupping partner for A ($A \oplus R \equiv_T 0'$). We use the finite extension method with a $0'$ oracle to construct R within a sequence of Π^0_1 classes. As we proceed, we have to remove R from the complements of some of the components of the universal A-Martin-Löf test; this can be done because A is now low for Martin-Löf randomness. At odd stages in the construction, we ensure that R does not compute D, and at even stages in the construction, we ensure that R is low. We make use of the fact that if we have a Π^0_1 class P and a $\Pi^0_1(A)$ class that contains only A-Martin-Löf random sets and a string τ with positive density in P, we can extend our τ to another string that is not in the $\Pi^0_1(A)$ class and still has positive density in P (in fact, a lower bound on this density can be found).

To see that $A \oplus R \geq_T 0'$, we note that, given R and A, we can determine which of the components of the universal A-Martin-Löf we have left. This calculation allows us to compute the settling time function for $0'$.

[7] We say that X has lower density zero in P if for all $\delta > 0$, there is an n such that the density of X in P, $d([X \upharpoonright n], P) = \mu([X \upharpoonright n] \cap P) * 2^{|\tau|}$, is less than δ.

Now we imagine removing the requirements that $A \leq_T 0'$ and $D \leq_T 0'$. Now our construction is computable in $A \oplus D \oplus 0'$, and we end up with a Martin-Löf random R such that $R \not\geq_T D$ and $A \oplus 0' \leq_T A \oplus R \leq_T A \oplus D \oplus 0'$. If we choose $D = 0'$, we have Theorem 3.6.

Later, Greenberg, Miller, Monin, and Turetsky proved another cupping result that gave them two more characterizations of $\mathcal{K}$ based on lowness for Ω [33]. In their aptly named paper "Two more characterizations of K-triviality," they prove and then make extensive use of the following result.

Theorem 3.7. [33] *Suppose that $A \not\leq_{LR} B$. Then for any set X, there is a B-Martin-Löf random Y such that $X \leq_T A \oplus Y$, and Y can even be made B-weakly 2-random.*

These new characterizations rely on the notion of one sequence being low for another:

Definition 3.8. [34] Let X be a random sequence. Then a sequence Y is *low for X* if X is Y-random.

Lowness for Chaitin's Ω has been carefully studied [1, 12, 51, 62, 63, 65]; however, the more general concept is useful as well, as we can see here. Recently, Yu developed another property based on lowness for a sequence which he called "absolutely low for X" [36] but was renamed in [33]:

Definition 3.9. Let X be a random sequence. A sequence A is *low for X preserving* if, for all sequences Y that are low for X, $A \oplus Y$ is low for X.

The first characterization is the following:

Theorem 3.10. [33] *A sequence A is in $\mathcal{K}$ if and only if it is low for Ω-preserving.*

Stephan and Yu had previously shown that every element of $\mathcal{K}$ is low for Ω preserving using the Low for Ω basis theorem [36], so we need only show the converse. We assume that A is low for X preserving for a Martin-Löf random X. We first argue that $A \leq_{LR} X$: if not, then the cupping result above gives us an X-random set Y such that $X \leq_T A \oplus Y$ and thus X is not $(Y \oplus A)$-Martin-Löf random. Van Lambalgen's Theorem can be used to show that X is therefore Y-random, though, so $A \leq_{LR} X$. Now we use a relativization of Kučera's classical result that every sequence is computable from an Martin-Löf random sequence [44] to observe that there is an X-Martin-Löf random Y such that $A \leq_T X \oplus Y$. Again, we can see that X is Y-Martin-Löf random (by van Lambalgen's Theorem) and X is $(Y \oplus A)$-Martin-Löf random (because A is low for X preserving). Furthermore, Y is A-Martin-Löf random because $A \leq_{LR} X$, and we can use van Lambalgen's Theorem relativized to A to see that $Y \oplus X$ is A-Martin-Löf random. However, since $A \leq_T Y \oplus X$, A is a base for randomness and thus $A \in \mathcal{K}$.

The second characterization involves LR reducibility.

Theorem 3.11. [33] *A sequence A is in $\mathcal{K}$ if and only if for all Y that are low for Ω, $Y \equiv_{LR} A \oplus Y$.*

The backwards direction is simple: we assume that for all Y that are low for Ω, $Y \equiv_{LR} A \oplus Y$. Since Ω is random with respect to $\emptyset$, we can see that $\emptyset \equiv_{LR} A \oplus \emptyset \equiv_{LR} A$.Therefore, A is low for Martin-Löf randomness.

The forward direction, once again, requires the cupping result above. We suppose that $A \in \mathcal{K}$, Y is low for Ω, and X is Y-random. Again, by relativizing Kučera's result, we can see that there is a Y-random W such that $W \oplus Y$ computes both X and Ω. We use the fact that if P is a nonempty

$\Pi^0_1(Y)$ class, then there is a sequence $S \in P$ that is low for W [12, 63] to find a set with PA degree relative to Y (one can find a $\Pi^0_1(Y)$ class containing only sequences that have PA degree relative to Y). We now have that W is S-random and that $Y \leq_T S$. A theorem of Miller and Yu allows us to see that X and Ω are also S-random [53]. Any PA degree that is low for Ω computes every element of $\mathcal{K}$ [65], so we have $A \leq_T S$. Now we are done: Since $A \oplus Y \leq_T S$ and X is S-random, X is $(A \oplus Y)$-random, and since our choice of X was arbitrary, we have $Y \equiv_{LR} A \oplus Y$.

We now focus more specifically on the computational weakness that the K-trivial reals exhibit. While noncupping is in fact a display of such weakness, we can further ask whether all elements of $\mathcal{K}$ are computable from a class of reals that are all themselves computationally weak.

Nies had already shown that $\mathcal{K}$ is a subclass of the superlow degrees and thus that it consists entirely of sequences that are computationally weak in the classical sense as well as the randomness sense [59]. Kučera showed in [43] that every Martin-Löf random sequence computable from $0'$ itself Turing computes a noncomputable c.e. set, and Hirschfeldt, Nies, and Stephan showed in [34] that Kučera's c.e. set must belong to $\mathcal{K}$. In fact, they also proved that the only c.e. sets computable from a Turing incomplete Martin-Löf random sequence were in $\mathcal{K}$. This suggested an extremely close relationship between enumerability and $\mathcal{K}$ and led to the following question by Stephan, later known as the ML-covering problem: Is every sequence in $\mathcal{K}$ computable from a Turing incomplete Martin-Löf random sequence?

This question has been answered affirmatively in a very strong way by Day and Miller in [11]: they proved that there was, in fact, a "weak" Martin-Löf random real that computed every element of $\mathcal{K}$. Their proof uses results from a large group of authors. The first of these results come from a paper by Bienvenu, Greenberg, Kučera, Nies, and Turetsky, in which they introduced the concept of Oberwolfach randomness, defined as follows.

Definition 3.12. [5] A test $\langle U_n \rangle = \langle W_{f(n)} \rangle$ is an *Oberwolfach test* if the following three conditions hold:

(1) the indexing function f is ω-c.e.,
(2) $\mu([U_n]) \leq 2^{-n}$ for all n, and
(3) for every n and every interval I of stages, if the version of U_n is constant on I, then there is at most one stage s in I at which the version of U_{n+1} changes.

A sequence is *Oberwolfach random* if it is not captured by any Oberwolfach test.

Clearly, every Martin-Löf test is an Oberwolfach test, but not vice versa. Oberwolfach randomness can also be characterized in terms of left-c.e. bounded tests; however, to do so, we first need to define the notion of a cost function.

Definition 3.13. A *cost function* is a computable function $c : \omega \times \omega \to \mathbb{Q}^{\geq 0}$. A cost function is further *monotonic* if $c(x+1, s) \leq c(x, s) \leq c(x, s+1)$ whenever $x < s$, and it is *additive* if $x < y < t$ implies that $c(x, t) = c(x, y) + c(y, t)$. The function $c(x) = \sup_s c(x, s)$ is called a *limit cost function.*

All cost functions in this paper will be assumed to be monotonic.

It should be noted here that any additive cost function is of the form $c_\beta(x, s) = \beta_s - \beta_x$ for some approximation $\langle \beta_i \rangle$ to a positive left-c.e. real β. We say that a computable approximation $\langle A_s \rangle$ obeys a cost function c if

$$\sum_{x,s} \{c(x, s) \mid x < s \text{ and } x \text{ is the least such that } A_{s-1}(x) \neq A_s(x)\} < \infty,$$

and we further say that a Δ^0_2 set A obeys a cost function if it has a computable approximation that does.

In this context, we use them to control the measure of the Σ^0_1 classes that serve as the components of a left-c.e. bounded test, which turns out to be equivalent to an Oberwolfach test.

Definition 3.14. Let c be a limit cost function. Then a uniformly Σ^0_1 test $\langle V_n \rangle$ is a *c-test* if $\mu([V_n]) \leq c(n)$ for all n, and a *left-c.e. bounded test* is a c-test in which c is the limit of an additive cost function.

Bienvenu, Greenberg, Kučera, Nies, and Turetsky showed that any sequence captured by an Oberwolfach test was also captured by a left-c.e. bounded test and vice versa [5]. This fact is necessary to prove the following theorem in [5]:

Theorem 3.15. *Let $X \in 2^\omega$. If X is Martin-Löf random but not Oberwolfach random, then X computes every element of $\mathcal{K}$.*

The proof of this theorem is straightforward. Nies proved in [59] that every K-trivial set is computable from a c.e. K-trivial set, so we need only show that our X computes every c.e. K-trivial set A. This X will fail a left-c.e. bounded test with an associated additive cost function c. We know that every c.e. K-trivial set A obeys every additive cost function; we note that while other proofs of this result have been given, there is a short direct proof in [5]. It is then enough to show that any set obeying this cost function is in $\mathcal{K}$. To do this, we show that if we have a c-test, then any Δ^0_2 set A that obeys c will be computable from every Martin-Löf random that is captured by this c-test. This is done by defining a functional Γ whose range consists of approximations of this A. This Γ is designed so that any Z captured by the c-test results in a total Γ^Z, and then it is shown that if such a Z is not captured by a particular Solovay test and thus may be Martin-Löf random, then this Z must compute our A.

The second necessary result from [5] involves density.

Theorem 3.16. *If X is Oberwolfach random, then X is a density-one point.*

We begin by showing that every left-c.e. martingale converges on every Oberwolfach random real. To do so, we define an interval test that captures X; rather than give the full definition of such a test, we will simply say that every real that is not Oberwolfach random is captured by a particular type of interval test. We can then convert the information about these martingales to a statement about dyadic density, and we find that for any effectively closed set $\mathcal{P}$ containing X, the dyadic density of $\mathcal{P}$ at X is 1. We can then convert this result to one about full density for irrational z; since all random points are irrational, this is enough. This lets us see that every Oberwolfach random real is a density-one point.

When we combine Theorems 3.15 and 3.16, we have the following (Theorem 1.1 in [5]):

Theorem 3.17. *A Martin-Löf random real that is not a density-one point computes every element of $\mathcal{K}$.*

The only remaining piece of the ML-covering problem is the following result from [11]:

Theorem 3.18. *There is a Δ^0_2 Martin-Löf random real that is a positive density point but not a density one point.*

To prove this, Day and Miller developed a new forcing notion. The conditions are pairs consisting of a finite binary string and a Π^0_1 class such that, given a nonempty Π^0_1 class P, two conditions hold for each $\langle \sigma, Q\rangle$:

(1) $[\sigma] \cap Q \neq \emptyset$ and
(2) there is a $\delta < \frac{1}{2}$ such that for all strings $\rho \succeq \delta$, if $[\rho] \cap Q \neq \emptyset$, then $\mu_\rho(Q) + \delta \geq \mu_\rho(P)$.

We begin with the condition $\langle \lambda, P\rangle$, where P is the aforementioned Π^0_1 class and define a sequence of forcing conditions $\langle p_i \rangle$ using a $\emptyset'$ oracle, guaranteeing us a Δ^0_2 result. For each Π^0_1 class S_e, we try to ensure that our set will belong to it; if the real we are building cannot be made to be a positive density point of S_e, we ensure that it will not be contained inside S_e. The desired Martin-Löf random will be the limit of the first coordinates of the p_is. In the end, our real will be contained in the correct Π^0_1 classes S. We are able to ensure that the limit of these first coordinates, X, is in fact an infinite sequence that is Martin-Löf random; we are further able to guarantee that the density of X with respect to P is less than $\frac{1}{2}$, which ensures that X is not a a density one point. We are further able to ensure that if X belongs to a Π^0_1 class S, then the density of X in that Π^0_1 class will be positive and thus that X is a positive density point.

This allows us to give a solution to the ML-covering problem at last. We note that since the point identified in Theorem 3.18 is a positive density point, it cannot be Turing complete.

Theorem 3.19. *There is a Turing incomplete Δ^0_2 Martin-Löf random sequence that computes every element of $\mathcal{K}$.*

4. Uniform relativization in randomness

Now we turn our attention to alternate relativizations. The results in Section 2 show us that the standard method of relativization for lowness and van Lambalgen's Theorem gives us intuitively correct results in the setting of Martin-Löf randomness but not Schnorr randomness or computable randomness. This leads very naturally to the following question: is there another approach to relativization that might result in the coincidence of lowness, triviality, and being a base and in van Lambalgen's Theorem holding for Schnorr and computable randomness? We begin with the earliest results in this area; after the initial investigations, the first steps towards a general theory of uniform relativization were made via investigations of van Lambalgen's Theorem.

4.1. **Initial results and van Lambalgen's Theorem.** Theorem 2.12 caused Franklin to suggest that Turing reducibility was not the right framework for Schnorr triviality and that a more natural framework might be the truth-table degrees [20]. She argued that Downey and Griffith's proof that the Schnorr trivial reals are closed downward in the tt-degrees provided further evidence that this was so [17]. Franklin and Stephan proceeded to show that this is, in fact, the case; they proved that two ways of being far from Schnorr random that seemed very different when Turing reducibility was considered actually coincide when tt-reducibility is used instead [27]. To relativize Schnorr randomness to a sequence A, instead of requiring that the martingale m be Turing computable from A, they used the stronger condition that $m \leq_{tt} A$. While they could have chosen to relativize the bound function as well, they argued that the end result is the same and opted for the simpler definition.

Definition 4.1. [27] A is tt-low for Schnorr randomness if for every Schnorr random set B, there is no martingale $d \leq_{tt} A$ and no computable function g such that for infinitely many n, $d(B\restriction g(n)) \geq n$.

This allows them to prove the following theorem that is far more parallel to the results for Martin-Löf randomness mentioned earlier than the results obtained using the standard relativization.

Theorem 4.2. [27] *Let A be a sequence. The following are equivalent:*

(1) *A is Schnorr trivial.*
(2) *A is tt-low for Schnorr randomness.*
(3) *The tt-degree of A is computably traceable (here, we trace the functions f that are tt-reducible to A instead of all $f \leq_T A$).*

The proof of this result involves a characterization of Schnorr triviality using computable probability distributions. This characterization is used to show that any function *tt*-computable from a Schnorr trivial sequence can be computably traced given an arbitrary bound function (in other words, that (1) implies (3)). We then translate this trace into information about martingales with the help of a technical lemma and show that A is *tt*-low for Schnorr randomness (so (3) implies (2)). At this point, we can carry out a highly calculational argument, once again using computable probability distributions, to show that (2) implies (1).

This approach can be generalized to define a general relativization of Schnorr randomness using *tt*-reducibility:

Definition 4.3. [27] A real A is truth-table Schnorr relative to another real B if there is no martingale $d \leq_{tt} B$ and no function $f \leq_{tt} B$ such that for infinitely many n, $d(A \upharpoonright f(n)) \geq n$.

Franklin and Stephan noted that since computations with *tt*-reductions only require us to know the oracle's values below the use, we can once again let f be a computable function in the definition above. This characterization allows us to prove a corollary that serves as another indication that the truth-table degrees are a very appropriate environment in which to study Schnorr randomness:

Corollary 4.4. [27] *The join of any two Schnorr trivial sequences is also Schnorr trivial, so the Schnorr trivial sequences form an ideal in the truth-table degrees.*

Miyabe used this work as a jumping-off point to make a study of alternate relativizations that more accurately reflected the way Schnorr and computable randomness work. As Franklin had done in [20], Miyabe argued in [54] that the fact that van Lambalgen's Theorem does not hold for Schnorr randomness suggests that the standard relativization for Schnorr randomness is not the most appropriate one. In [54], Miyabe (and then Miyabe and Rute in [58]) introduced variants on Schnorr randomness and computable randomness in which the relativizations were based on truth-table reductions rather than Turing reductions. Miyabe called the Schnorr randomness version "truth-table Schnorr randomness" in [54], and he and Rute changed the terminology to "uniformly relative Schnorr randomness" in [58]. We begin with the definitions required to understand his approach. The first are some fundamental definitions from computable analysis as given in [58].

Definition 4.5. A *code* for an open subset of 2^ω is a sequence of basic open sets of 2^ω whose union is the original open subset, and an open set is *effectively open* if it has a computable code.

A sequence $\langle q_n \rangle$ of rationals is *fast Cauchy* if, whenever $n \geq m$, we have $|q_n - q_m| \leq 2^{-m}$.

Definition 4.6. [58] We say that a function $f : \omega \to \omega$ *encodes a Schnorr test* $\langle U_n \rangle$ if it encodes both

(1) a list of basic open sets whose union is U_n for each n and

(2) a Cauchy sequence of rationals $\langle q_n \rangle$ converging to each measure $\mu(U_n)$ such that for all $n \geq m$, $|q_n - q_m| \leq 2^{-m}$.

A *uniform Schnorr test* is a total computable map $\Phi : 2^\omega \to \omega^\omega$ such that $\Phi(X)$ encodes a Schnorr test for every X.

It may be easier to think of a uniform Schnorr test as Miyabe did in [56]: a computable function $f : 2^\omega \times \omega \to \mathcal{O}$ such that $\langle X, i \rangle \mapsto \mu(f(X,i))$ is computable and $\mu(f(X,i)) \leq \frac{1}{2^i}$ for all n. In any case, now we can define a uniform version of Schnorr randomness:

Definition 4.7. [54, 58] A real A is *uniformly Schnorr random relative to a real B* if there is no uniform Schnorr test $\langle U_n^X \rangle$ such that $A \in \cap_n U_n^B$.

Miyabe notes in [54] that the basic facts that one would expect to be true of uniform Schnorr randomness are, in fact, true: that a sequence is uniformly Schnorr random exactly when it is Schnorr random (Prop. 4.4), that if a sequence is X-Schnorr random, then it is uniformly Schnorr random relative to X (Prop. 4.5), and that there is no universal uniform Schnorr test relative to X for any X (Prop. 4.9).

In [54], Miyabe showed that Franklin and Stephan's notion of truth-table Schnorr randomness in [27] is equivalent to his notion of uniformly relative Schnorr randomness; here, we present the more complete proof present in [58] using a uniform relativization for the martingale approach:

Definition 4.8. The sequence $\langle d^X \rangle$ is a *uniform martingale test* if it is given by a computable map $\Phi : 2^\omega \to \omega^\omega$ such that each $\Phi(X)$ encodes a martingale.

Again, it may be easier to think of a uniform martingale test as Miyabe defined it in [56]: a computable function $m : 2^\omega \times 2^{<\omega} \to \mathbb{R}^{>0}$ such that $m^Z = m(Z, -)$ is a martingale for every sequence Z.

Proposition 4.9. *A set A is uniformly Schnorr random relative to B if and only if A is truth-table Schnorr random relative to B.*

Proof. We begin by showing that if d is a martingale and $f : \mathbb{N} \to \mathbb{N}$ is a function, both *tt*-below A, then there is a uniform martingale test $\langle \hat{d}^X \rangle$ and a uniform function $\langle \hat{f} \rangle$ such that $\hat{d}^A = d$ and $\hat{f}^A = f$. (The converse to this statement, which will also be used, is trivial.)

The result for f is easy; we simply observe that there is a total functional Ψ such that $\Psi(A) = f$ and define $\hat{f}^X$ to be $\Psi(X)$. The result for d requires some bookkeeping and care and will be dependent on the choice of code for d. We suppose that we have a total computable functional Φ such that $\Phi(A, \sigma)$ encodes the fast-Cauchy code for $d(\sigma)$ and define $d^X(\sigma)$ to be the real coded by $\Phi(X, \sigma)$. We note that it is not a given that $\Phi(X, \sigma)$ will be a Cauchy code for a nonnegative real for each pair (X, σ); to guarantee this, we take the original sequence of rationals given by $\Phi(X, \sigma)$, replace all negative entries with 0, and then remove the terms that would cause the sequence to either converge too slowly or diverge. Furthermore, we must guarantee that $\hat{d}^X$ is truly a martingale for every X. To do so, we define it recursively from d^X: we let $\hat{d}^X(\langle\rangle) = d^X(\langle\rangle)$, set $\hat{d}^X(\sigma 0)$ equal to the minimum of $d^X(\sigma 0)$ and $2\hat{d}^X(\sigma)$, and define $\hat{d}^X(\sigma 1)$ so that the basic martingale fairness condition is satisfied. We can note that since d^A was initially a martingale, then $\hat{d}^A$ is necessarily d^A and that the code for $\hat{d}^X$ is uniformly computable from the code for d^X.

This result lets us see that truth-table Schnorr randomness is equivalent to uniform relativization given the martingale definition; the standard proofs allow us to effectively convert a uniform martingale test to a Schnorr test covering the same points. □

In Section 4 of [54], Miyabe claimed a proof that van Lambalgen's Theorem holds for uniformly relative Schnorr randomness that closely parallels the standard proof for Martin-Löf randomness mentioned previously. However, an unwarranted assumption was made in this proof: it had been assumed that, given a Schnorr test relativized to A, $\langle V_i^A \rangle$, the measure of V_i^A could be computed uniformly from A, a code for V_i, and the measure of V_i. This is corrected in [58] with a new proof using integral tests. Furthermore, the particular assumption that had been made sheds light on the reason that van Lambalgen's Theorem does not hold for Schnorr randomness with the standard relativization: this type of computation may not allow us to distinguish between two sets with very different measures provided that the difference in measure results from a small "gap" [58]. At this point, we must formally define integral tests to proceed.

Definition 4.10. An *integral test* is a lower semicomputable function $t : 2^\omega \to [0, \infty]$ such that $\int t \; d\mu < \infty$. If $\int t \; d\mu$ is computable, then t is a *Schnorr integral test*, and a *uniform Schnorr integral test* is a total computable map $\Phi : 2^\omega \to \omega^\omega$ such that $\Phi(X)$ encodes a Schnorr integral test for every X.

For ease of notation, we may also call $\langle t^X \rangle$ a uniform Schnorr integral test if it is given by a map Φ as above.

Miyabe had already shown that the following is true:

Proposition 4.11. [55] *A real A is Schnorr random if and only if there is no Schnorr integral test t such that $t(A) = \infty$.*

As mentioned above, Miyabe and Rute provide a remarkably short and elegant proof of the "hard" direction of van Lambalgen's Theorem for standard Schnorr randomness using integral tests [58]. They then show that the following version of van Lambalgen's Theorem for uniformly relative Schnorr randomness holds.

Theorem 4.12. [58] *$A \oplus B$ is Schnorr random if and only if A is Schnorr random and B is Schnorr random uniformly relative to A.*

The proof of the "easy" direction is given in [54] and is a modification of the proof of this direction for Martin-Löf randomness. To prove the "hard" direction, we assume that A is Schnorr random and $A \oplus B$ is not and take a Schnorr integral test t witnessing that $A \oplus B$ is not Schnorr random. For each X, we set $t^X(Y) = t(X \oplus Y)$. This $\langle t^X \rangle$ may not be a uniform Schnorr integral test itself, but we can construct a uniform Schnorr integral test $\langle \hat{t}^X \rangle$ from it by enumerating only part of t^X so that $\hat{t}^A = t^A$. This test will witness that B is not Schnorr random uniformly relative to A.

Miyabe and Rute also present a uniform relativization for computable randomness using the concept of a uniform martingale test presented earlier:

Definition 4.13. A sequence A is *computably random uniformly relative to B* if there is no uniform martingale test $\langle d^X \rangle$ such that d^B succeeds on A.

While they do not prove a theorem quite as strong as van Lambalgen's Theorem for uniformly relative computable randomness, they are able to show a slightly weaker theorem using a relatively traditional martingale approach.[8]

[8]In [54], the author claims a proof that van Lambalgen's Theorem holds for a uniform version of computable randomness; however, this was retracted in [58] and this weaker result is given instead.

Theorem 4.14. [58] *$A \oplus B$ is computably random if and only if A is computably random uniformly relative to B and B is computably random uniformly relative to A.*

To show the "hard" direction, we take a martingale m witnessing that $A \oplus B$ is not computably random and use it as we did in the proof of Theorem 4.12 to construct two martingales, m_1^B and m_2^A, that, for some set X, mimic the behavior of m on $X \oplus B$ and $A \oplus X$, respectively. Then we show that either m_1^B succeeds on A or m_2^A succeeds on B. The "easy" direction is proven in [54] using the test definition of computable randomness and showing that if $A \oplus B$ is computably random, then B must be computably random uniformly relative to A. The proof proceeds by contradiction; we assume that B is not computably random uniformly relative to A and produce a test witnessing that $A \oplus B$ could not be computably random.

Kihara and Miyabe later turned their attention to Kurtz randomness [38]. Here, we let $\mathcal{O}$ be the class of open sets on 2^ω.[9]

Definition 4.15. A *uniform Kurtz test* is a total computable function $f : 2^\omega \to \mathcal{O}$ such that $\mu(f(X)) = 1$ for all $X \in 2^\omega$. A sequence B is Kurtz random uniformly relative to A if $B \in f(A)$ for every uniform Kurtz test f.

They also give definitions in terms of the standard test approach as well as in terms of martingales and machines [38]; we present the test approach here.

Definition 4.16. A *uniform Kurtz null test* is a computable function $g : 2^\omega \times \omega \to (2^{<\omega})^{<\omega}$ such that for every sequence X and every n, $\mu([g(X, n)]) \leq 2^{-n}$.

They used these definitions to give the following characterizations of one set being (not) Kurtz random uniformly relative to another; the proofs generally involve a careful conversion of one type of test to another.

Theorem 4.17 (Prop. 3.5, [38])**.** *The following are equivalent:*

1. *A is not Kurtz random uniformly relative to B.*
2. *For some uniform Kurtz test f, $A \notin f(B)$.*
3. *For some uniform Kurtz null test g, $A \in \cap_n [g(B, n)]$.*
4. *There is a computable function $d : 2^\omega \times 2^{<\omega} \to \mathbb{R}^+$ and a computable order function h such that $d(X, -)$ is a martingale for every X and $d(A\restriction n) > h(n)$ for all n.*
5. *There are an oracle prefix-free machine M and a computable function h such that $X \mapsto \mu(dom(M^X))$ is computable and $K_{M^B}(A\restriction h(n)) < h(n) - n$ for all n.*

Now we can discuss van Lambalgen's Theorem in the context of uniform Kurtz randomness. Franklin and Stephan had shown that the "hard direction" of van Lambalgen's Theorem holds for Kurtz randomness, but not the "easy direction" [28]. In [38], Kihara and Miyabe show that exactly the opposite is true for uniform Kurtz randomness: the "easy direction" holds, but not the "hard direction."

Theorem 4.18. *If $A \oplus B$ is Kurtz random, then B is Kurtz random uniformly relative to A.*

The proof is quite short and proceeds by contradiction. We assume that B is not Kurtz random uniformly relative to A, take a uniform Kurtz test f witnessing this fact, and use this to define a c.e. set $U = \{X \oplus Y \mid Y \in f(X)\}$. The measure of U will be 1, and $A \oplus B$ will not be an element of U, so $A \oplus B$ cannot be Kurtz random.

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[9]Kihara and Miyabe call this τ, but I have changed the notation since I reserve τ for finite binary strings.

Theorem 4.19. *There are sequences A and B that are uniformly Kurtz random relative to each other such that $A \oplus B$ is not itself Kurtz random.*

We begin by taking an enumeration $\langle \Phi_i \rangle$ of all uniform Kurtz tests and note that we can think of such a test Φ_i as a map $\sigma \mapsto \Phi_i(\sigma)$ such that $\Phi_i(X) = \cup_{\sigma \prec X} \Phi_i(\sigma)$. The set $\Phi_i(\sigma 0^\omega)$ is conull and will therefore be dense. Our sequences A and B will be constructed in stages: at stage s, we define two finite sequences, α_s and β_s, with the same length.

At even stages $2i$, we find a string $\beta \succ \beta_s$ and an $m \in \omega$ such that $[\beta] \subseteq \Phi_i(\alpha_s 0^m)$. We define $\alpha_{s+1} = \alpha_s 0^m$, and we define β_{s+1} to be the string resulting from extending β with 0s to the length of α_{s+1}; at odd stages $2i+1$, we reverse the roles of α_s and β_s. We then set $A = \cup_s \alpha_s$ and $B = \cup_s \beta_s$.

This will result in either A or B being 0^ω, so $A \oplus B$ cannot be Kurtz random. However, they have both been constructed to be uniformly Kurtz random relative to the other, so we have our result.

A word should be said at this point about Martin-Löf randomness. In [58], Miyabe and Rute point out that Martin-Löf randomness relative to an oracle is equivalent to Martin-Löf randomness uniformly relative to an oracle (and that the same relationship holds for Kolmogorov-Loveland randomness, in fact), so there is no pressing need to adopt or reject uniform relativization in this case; the same results will hold regardless.

4.2. **Lowness, etc. for uniform relativization.** Now we turn to a more general discussion of randomness under uniform relativizations. In [54], Miyabe also continued Franklin and Stephan's study of tt-lowness for Schnorr randomness using a uniformization of computable measure machines:

Definition 4.20. [54] A prefix-free machine M is a *tt-reducible measure machine* if there is a Turing functional Φ such that $\Omega_M^X = \Phi^X$ for all sequences X.

Later, Miyabe will call these *uniformly computable measure machines* [56], however, we will use this notation here.

It is straightforward to show that a sequence A is X-tt-Schnorr random if and only if for all truth-table reducible measure machines M, there is a constant c such that for all n,

$$K_M^X(A \restriction n) > n - c.$$

This is done by converting machines to Schnorr tests and vice versa in the standard way in Theorem 7.1.15 of [19]; the traditional construction is sufficiently uniform.

This allows Miyabe to introduce the definition of lowness for tt-reducible measure machines formally [54].

Definition 4.21. A sequence A is *low for tt-reducible measure machines* if for each A-tt-reducible measure machine M, there is a tt-reducible measure machine N such that for some constant c and for all strings σ,

$$K_N(\sigma) \leq K_M(\sigma) + c.$$

Since there is no universal uniform Schnorr test for any oracle, it is not obvious that lowness for truth-table reducible measure machines corresponds to lowness for tt-Schnorr randomness; Miyabe proves that this is in fact true.

Theorem 4.22. [54] *A sequence is low for tt-reducible measure machines if its truth-table degree is computably traceable.*

The forward direction is true because lowness for tt-reducible measure machines implies lowness for tt-Schnorr randomness; the backward direction is a modification of the proof for the corresponding result for the standard relativization as found in [13].

Later, Kihara and Miyabe would consider lowness for uniform Kurtz randomness [38]. Here, the class of reals that are low for the uniform notion differs sharply from the class of those that are low for the standard relativization. As mentioned in Section 2, Greenberg and Miller showed that the sequences that are low for Kurtz randomness are those that are hyperimmune free and cannot compute a diagonally noncomputable function [32]. However, hyperimmune-freeness is not a useful concept in the truth-table degrees, so no parallel result can be obtained in the current context. We begin here with the basic definition of lowness for a uniform Kurtz test.

Definition 4.23. Let A be a sequence. A is *low for uniform Kurtz tests* if $f(A)$ contains a Kurtz test for every uniform Kurtz test f.

In order to discuss this, Kihara and Miyabe had to define a new traceability notion:

Definition 4.24. A computable trace $\langle D_n \rangle$ *Kurtz-traces* a function f if there is a strictly increasing computable sequence $\langle \ell_n \rangle$ of natural numbers such that for all k, there is some $n \in [\ell_k, \ell_{k+1})$ such that $f(n) \in D_n$.

A set A is *Kurtz tt-traceable* if there is a computable order function p such that for every $f \leq_{tt} A$, some computable trace with bound p Kurtz-traces f.

This definition allows Kihara and Miyabe to characterize lowness for uniform Kurtz randomness (and uniform Kurtz tests) in terms of traceability.

Theorem 4.25. [38] *Let A be a sequence. The following are equivalent.*

(1) *A is low for uniform Kurtz tests.*
(2) *A is low for uniform Kurtz randomness.*
(3) *A does not tt-compute an infinite subset of any Kurtz random.*
(4) *A is Kurtz tt-traceable.*

Proof. It is clear that (1) implies (2). To complete the proof, we need a new way of describing subsets of ω: as strings σ in $\omega^{\leq\omega}$ such that $\sigma(n) < \sigma(n+1)$ for each n, so $\sigma(n)$ is the n^{th} element of the set it defines (note that σ can be infinite or finite). This allows us to define $\hat{P}^{\sigma}$ to be $\{X \in 2^{\omega} \mid \mathrm{rng}(\sigma) \subseteq X\}$. Throughout this proof, we adopt the authors' convention of writing B^* for the element of $\omega^{\leq\omega}$ that corresponds to the set $B \subseteq \omega$.

Kihara and Miyabe then prove that a set A will tt-compute an infinite subset of a Kurtz random set if and only if there is some infinite $B \leq_{tt} A$ such that the class $\hat{P}^{B^*}$ contains a Kurtz random set. Then they show that if B is infinite and $B \leq_{tt} A$, then $\hat{P}^{B^*}$ is a Kurtz null test uniformly relative to A. The former result is relatively straightforward; the latter can be seen by supposing that Ψ witnesses $B \leq_{tt} A$ and using Ψ to build a map Φ such that Φ^X defines an set $B(X)$ for every X such that whenever $B(X)$ is infinite, then $\hat{P}^{B(X)^*}$ is null. Therefore, $X \mapsto \hat{P}^{B(X)^*}$ is a uniform Kurtz null test, and $\hat{P}^{B(A)^*}$ is a Kurtz null test uniformly relative to A.

This makes the proof of (2) implies (3) extremely straightforward: if A is low for Kurtz randomness, then for every $B \leq_{tt} A$, $\hat{P}^{B^*}$ will be a Kurtz null test uniformly relative to A, and since A is low for Kurtz randomness, $\hat{P}^{B^*}$ can't contain a Kurtz random sequence. By the result above, this means that A cannot tt-compute an infinite subset of a Kurtz random set.

The proof that (3) implies (4) is heavily based on Greenberg and Miller's work concerning lowness for Kurtz randomness and involves a somewhat simpler version of the svelte trees from [32].

Finally, the proof that (4) implies (1) relies on a notion of dimension: Kihara and Miyabe define a set A to be *Kurtz h-dimensional measure 0* if there is a computable sequence $\langle C_n \rangle$ of finite sets of strings such that $A \in [C_n]$ and $\sum_{\sigma \in C_n} 2^{-h(|\sigma|)} \leq 2^{-n}$ for every n; here, h is simply a real-valued function. Given a computable trace for A with a certain order function and strings of known lengths in each component, we can define a sequence $\langle C_n \rangle$ as above to see that A is Kurtz h-dimensional zero for any computable order h. Now we can suppose that A is Kurtz h-dimensional zero. We begin by taking a Kurtz null test uniformly relative to A and use the sequence $\langle C_n \rangle$ to construct a Kurtz null test whose intersection contains that of the Kurtz null test uniformly relative to A, proving the result. □

This allows us to see that lowness for uniform Kurtz randomness and lowness for Kurtz randomness do not coincide [38]. A set that is low for uniform Schnorr randomness must be low for uniform Kurtz randomness as well, since the sets that are low for uniform Schnorr randomness are those that are computably tt-traceable [27], and these traces will Kurtz-tt-trace the set as well. However, Franklin has shown that there is a 1-generic Schnorr trivial set [23], and such a set must be low for uniform Schnorr randomness [27]. However, all 1-generic sets are Kurtz random [46], so this set must be low for uniform Kurtz randomness but not low for Kurtz randomness.

Kihara and Miyabe have also considered lowness for uniform relativization in the context of pairs of randomness notions:

Definition 4.26. [39] $\mathrm{Low}^\bigstar(\mathcal{C}, \mathcal{D})$ is the set of sequences A such that $\mathcal{C}$ is contained in the class of sequences that are in $\mathcal{D}$ uniformly relative to A.

They characterize $\mathrm{Low}^\bigstar(\mathcal{C}, \mathcal{D})$ for several pairs of randomness notions where $\mathcal{C}$ and $\mathcal{D}$ range over ML, $\mathsf{Schnorr}$, and Kurtz in terms of various forms of traceability and anticomplexity [39].

The relevant traceability notions are variants on computable traceability and c.e. traceability. If we write "tt-traceability" rather than simply "traceability," we mean that that every $f \leq_{tt} A$ must be traceable rather than every $f \leq_T A$. A function is *computably often* (c.o.) traceable if the Kurtz traceability condition from [38] holds, though the trace in question may be c.e. or computable. Finally, a function is *infinitely often* (i.o.) traceable if $f(n)$ is an element of the trace in question for infinitely many n.

Their proofs rely heavily on previous characterizations of $\mathrm{Low}(\mathcal{C}, \mathcal{D})$ and technical means of uniformizing the proofs of these characterizations. Their results are summarized below:

Theorem 4.27. [39] *Let A be a sequence.*

1. *A is in $Low^\bigstar(\mathsf{ML}, \mathsf{Schnorr})$ if and only if A is c.e. tt-traceable.*
2. *A is in $Low^\bigstar(\mathsf{Schnorr})$ if and only if A is computably tt-traceable.*
3. *A is in $Low^\bigstar(\mathsf{ML}, \mathsf{Kurtz})$ if and only if A is c.e. i.o. tt-traceable.*
4. *A is in $Low^\bigstar(\mathsf{Schnorr}, \mathsf{Kurtz})$ if and only if A is computably i.o. tt-traceable.*
5. *A is in $Low^\bigstar(\mathsf{Kurtz})$ if and only if A is computably c.o. tt-traceable.*

The first of these results is proven using a covering property of Bienvenu and Miller [7] and several technical lemmas; the second is inherent in Franklin and Stephan [27]. The third is proven using a sequence of lemmas and modifications of the proof of Theorem 6.1 in [38] and the proof of Theorem 8.10.2 in [19]. The fourth is provable using the techniques in [32], and the fifth is our Theorem 4.25.

A series of corresponding results involving anticomplexity is also obtained through appeals to [35] and [40]; we will simply define the anticomplexity notions and then present the results.

Definition 4.28. Let A be a sequence and let h be an order function.

(1) A is *complex* if there is some order function h such that $C(A\restriction h(n)) \geq n$ for all n [40].
(2) A is *anticomplex* if for every order function h, $C(A\restriction h(n)) \leq n$ for all n [24].
(3) A is *totally complex* if there is some order function h such that for every machine M with total domain, $C_M(A\restriction h(n)) \geq n$ for all n [39].
(4) A is *totally anticomplex* if for every order function h, there is a machine M with total domain such that $C_M(A\restriction h(n)) \leq n$ for almost all n [39].
(5) A is *totally c.o. anticomplex* if for every order function h, there is a machine M with total domain and an order function ℓ such that for every n, $C_M(A \restriction h(n)) \leq n$ for some $n \in [\ell(k), \ell(k+1))$ [39].

Theorem 4.29. [39] *Let A be a sequence.*

(1) *A is in $Low^\bigstar(\mathsf{ML}, \mathsf{Schnorr})$ if and only if A is anticomplex.*
(2) *A is in $Low^\bigstar(\mathsf{Schnorr})$ if and only if A is totally anticomplex.*
(3) *A is in $Low^\bigstar(\mathsf{ML}, \mathsf{Kurtz})$ if and only if A is not complex.*
(4) *A is in $Low^\bigstar(\mathsf{Schnorr}, \mathsf{Kurtz})$ if and only if A is not totally complex.*
(5) *A is in $Low^\bigstar(\mathsf{Kurtz})$ if and only if A is totally c.o. anticomplex.*

However, lowness and triviality are not the only such concepts for which a uniform relativization has been considered. Franklin and Stephan also considered a truth table version of bases: one in which A is a *tt*-base for Schnorr randomness if and only if there is a $B \geq_{tt} A$ such that B is A-Schnorr random. While they showed that this is not equivalent to Schnorr triviality, as hoped, they do show in [27] that it is sufficient for Schnorr triviality.

Miyabe took this study a step farther by defining bases for uniform Schnorr tests and uniformly computable martingales:

Definition 4.30. [56] Let A be a sequence.

(1) A is a *base for uniform Schnorr tests* if for every computable martingale m uniformly relative to A, there is a $B \geq_{tt} A$ that is Schnorr random uniformly relative to A for m.
(2) A is a *base for uniformly computable martingales* if for every computable martingale m uniformly relative to A, there is a $B \geq_{tt} A$ that is computably random uniformly relative to A for m.

Miyabe then goes on to prove that these notions are equivalent to Schnorr triviality:

Theorem 4.31. [56] *Let A be a sequence. The following are equivalent:*

(1) *A is Schnorr trivial.*
(2) *A is a base for uniformly computable martingales.*
(3) *A is a base for uniform Schnorr tests.*

It is clear that (2) implies (3), and we can see that (3) implies (1) by examining the proof of Proposition 6.1 in [27]: we can note that if $A \leq_{tt} B$, either A is Schnorr trivial or B is not Schnorr random uniformly relative to A. To prove that (1) implies (2), we construct our set B using the Space Lemma [49] and the fact that A must be computably tt-traceable.

Therefore, when we consider Schnorr randomness under the standard relativization, the lowness notions (lowness for Schnorr randomness, lowness for Schnorr tests, and lowness for computable measure machines) all coincide, but triviality and being a base coincide neither with the lowness notions nor with each other. However, under uniform relativization, all these notions coincide. This lends incredible strength to the argument that uniform relativization is the most reasonable approach to relativizing definitions related to Schnorr randomness and possibly any randomness notion.

4.3. **Reducibilities.** This work has also given rise to some reducibilities related to uniform relativizations, most of which are connected to Schnorr randomness. Their predecessor is Schnorr reducibility, $\leq_{Sch}$, introduced by Downey and Griffiths (and Laforte) in [14] and [17].

Definition 4.32. *A is Schnorr reducible to B (written as $A \leq_{Sch} B$) if for each computable measure machine M, there is a computable measure machine N and a constant c such that for every n,*

$$K_N(A{\restriction}n) \leq K_M(B{\restriction}n) + c.$$

Clearly, a set is Schnorr trivial if it is Schnorr reducible to $\emptyset$.

Later, in [27], Franklin and Stephan introduced the reducibility $\leq_{snr}$:

Definition 4.33. *$A \leq_{snr} B$ if and only if there is a computable function h such that*

$$(\forall f \leq_{tt} A)(\exists g \leq_{tt} B)(\forall n)(\exists m \leq h(n))[f(n) = g(m)].$$

Their characterization of a Schnorr trivial set in terms of the traceability of functions tt-below it lets them see automatically that $A \leq_{snr} \emptyset$ if and only if A is Schnorr trivial. This allowed them to characterize the Schnorr trivial sequences in a way reminiscent of the definition of a base:

Theorem 4.34. *A sequence A is Schnorr trivial if and only if there is a sequence B such that B is truth-table Schnorr random relative to A and $A \leq_{snr} B$.*

The first half of the proof is easy: If A is Schnorr trivial, then $A \leq_{snr} B$ for every set B. For the second half, we take a strictly increasing computable function and the computable bound h from the snr-reduction. We then use the use function for the tt-reduction that computes a set of up to $h(n)$ strings that includes $A{\restriction}f(n)$ for each n and a martingale based on this use function to list $4^n h(n)$ strings that include $A{\restriction}f(n)$.

Miyabe introduced a new reducibility similar to $\leq_{Sch}$ but using an additional order function in [56].[10]

Definition 4.35. *A is a weakly decidable prefix-free machine reducible to B (written $A \leq_{wdm} B$) if for every decidable prefix-free machine M and order function g, there is a decidable prefix-free machine N and a constant c such that for every n,*

$$K_N(A{\restriction}n) \leq K_M(B{\restriction}n) + g(n) + c.$$

While the reducibility $\leq_{wdm}$ is actually identical to $\leq_{Sch}$, Miyabe gives an independent proof in [56] that a sequence is Schnorr trivial exactly when it is wdm-below $\emptyset$. If $A \leq_{wdm} \emptyset$ and h is an order function, then an order function g such that $(g \circ h)(n) \leq n/2$ can be found. Now we can find

[10]In [56], a *decidable* prefix-free machine is simply one with computable domain

a decidable prefix-free machine M such that for some c, $K_M(h(n)) \leq 2\log n + c$ for all n; since $A \leq_{wdm} \emptyset$, we can find a decidable prefix-free machine N such that $K_N(A{\upharpoonright}h(n)) < n$ for all n. Since the Schnorr trivial sets are precisely those that are not totally i.o. complex [35], we are done with the backwards direction.[11]

To prove the forwards direction, we note that if A is Schnorr trivial, then it is computably tt-traceable. Now we take a decidable prefix-free machine M and an order function g and choose L to be a decidable prefix-free machine such that $K_L(0^n) \leq 2\log n$. If we set $f(n) = A{\upharpoonright}n$, then $f \leq_{tt} A$, and we can find a computable trace $\langle T_n \rangle$ for f. Now we can define a decidable prefix-free machine N from M, L , and $\langle T_n \rangle$ witnessing that $A \leq_{wdm} \emptyset$.

Miyabe then goes on to discuss an analogue to $\leq_{LR}$ (and therefore $\leq_{LK}$) for Schnorr randomness called $\leq_{LUS}$.

Theorem 4.36. [56] *Let A and B be sequences. Then the following are equivalent, and when any of them hold, we say that $A \leq_{LUS} B$.*

1. *Every set that is Schnorr random uniformly relative to B is Schnorr random uniformly relative to A.*
2. *Every Schnorr test uniformly relative to A is covered by a Schnorr test uniformly relative to B.*
3. *Given any uniformly computable measure machine M, there is a uniformly computable measure machine N such that for some c,*
$$K_{N^B}(\sigma) \leq K_{M^A}(\sigma) + c$$
for every σ.
4. *Given any strictly bounded and uniformly Schnorr function $g : 2^\omega \to \mathcal{O}$, there is a strictly bounded and uniformly Schnorr function $h : 2^\omega \to \mathcal{O}$ such that $g(A) \subseteq h(B)$.*
5. *Given a computable function $f : 2^\omega \times \mathbb{N} \to \mathbb{R}^+$ such that $X \mapsto \sum_{n=0}^{\infty} f(X, n)$ is computable, there is a computable function $g : 2^\omega \times \mathbb{N} \to \mathbb{R}^+$ such that $X \mapsto \sum_{n=0}^{\infty} g(X, n)$ is computable and $f(A, n) \leq g(B, n)$ for all n.*

This proof is extremely long, so only the basic ideas in it will be sketched out. To prove that (3) implies (2), we note that from a uniform Schnorr test, we can generate a uniformly computable measure machine M such that the set of all X with low complexity relative to that machine with oracle A contains the intersection of this test with respect to the oracle A. By our hypothesis, there is a uniformly computable measure machine that assigns lower complexities with respect to B than M does with respect to A, and we can use this to construct a Schnorr test uniformly relative to B whose intersection contains that of the original Schnorr test uniformly relative to A.

The proof that (2) implies (1) is entirely straightforward. The proofs that (1) implies (4), (4) implies (5), and (5) implies (1) are all based on Bienvenu and Miller's work on open covers [7].

In [57], Miyabe continues investigating reducibilities for Schnorr randomness parallel to well-studied reducibilities for Martin-Löf randomness. In addition to the classical C-reducibility, he considers other reducibilities more closely tied to relativization such as Nies's $\leq_{CT}$ ([61], Ex. 8.4.21) and a version of vL-reducibility tied to Schnorr randomness instead of Martin-Löf randomness, $\leq_{vLS}$.

The reducibility $\leq_{CT}$ is based on computable traceability and is defined as follows:

[11] A set is totally i.o. complex if there is a computable function g such that for all machines M with total domain, there are infinitely many n such that $C_M(A{\upharpoonright}g(n)) \geq n$.

Definition 4.37. Given sequences A and B, we say that $A \leq_{CT} B$ if there is an order function h such that for every $f \leq_T A$, there is a $p \leq_T B$ such that for every n, $f(n) \in D_{p(n)}$ and $|D_{p(n)}| \leq h(n)$.

We note that $A \leq_{CT} \emptyset$ precisely when A is computably traceable. Miyabe generalized this to the situation where $A \leq_{CT} B$:

Theorem 4.38. [57] *Given sequences A and B, $A \leq_{CT} B$ if and only if every Schnorr random sequence relative to B is Schnorr random relative to A.*

To prove this, we first suppose that $A \leq_{CT} B$. Our goal is to demonstrate that every A-Schnorr test is covered by a B-Schnorr test; this is done by constructing an open set relative to B that is a union of open sets relative to all the elements of the B-trace for A as in Theorem 2 in [71]; the traceability condition will result in only a very few possible oracles, and the resulting set will be small enough.

Now let us suppose that every Schnorr random sequence relative to B is Schnorr random relative to A. This direction requires the "open covers" technique of Bienvenu and Miller [7]; Miyabe makes ample use of the results from this paper and a judicious application of the Kraft-Chaitin Theorem.

Finally, Miyabe addresses the Schnorr equivalent of $\leq_{vL}$. His definition of $\leq_{vLS}$ in [57] is identical to that of $\leq_{vL}$ in [53], but with "Martin-Löf" replaced by "Schnorr":

Definition 4.39. Given sequences A and B, $A \leq_{vLS} B$ if for all $X \in 2^\omega$, whenever $A \oplus X$ is Schnorr random, $B \oplus X$ is Schnorr random.

We note that van Lambalgen's Theorem shows us that if A and B are both Martin-Löf random, then $A \leq_{vL} B$ if and only if $B \leq_{LR} A$. Similarly, van Lambalgen's Theorem for Schnorr randomness lets us see that $A \leq_{vLS} B$ if and only if $B \leq_{LUS} A$ for Schnorr randoms A and B. Now we can relate $A \leq_{vLS}$ to our Schnorr reducibility $\leq_{Sch}$: Miyabe has shown that if $A \leq_{Sch} B$, then $A \leq_{vLS} B$ [57]. This result follows from the following theorem; we note that $B\restriction n$ represents the natural number k such that $B\restriction n$ is the k^{th} string in the lexicographic order on $2^{<\omega}$.

Theorem 4.40. *Given sequences A and B, $A \oplus B$ is Schnorr random if and only if for every computable measure machine M, there is a c such that*

$$K_M(A\restriction(B\restriction n)) \geq (B\restriction n) + n - c. \tag{1}$$

The proof given in [57] is a modification of Theorem 5.1 in [53]. We begin by assuming that $A \oplus B$ is Schnorr random and converting it to a variant, $A\widehat{\oplus}B$, in which multiple bits of A are inserted between the bits of B in a particular computable way. $A \oplus B$ is Schnorr random exactly when $A\widehat{\oplus}B$ is, and for any computable measure machine M, we can find an auxiliary computable measure machine N that lets us show that Equation 1 holds. To prove the other direction, we assume that $A \oplus B$ is not Schnorr random. Once again, we use the equivalence between $A \oplus B$ and $A\widehat{\oplus}B$ and, for an arbitrary computable measure machine M, build an auxiliary computable measure machine N that lets us show that there is no c such that $K_M(A\restriction(B\restriction n))$ is always at least as large as $(Z\restriction n) + n - c$.

5. Conclusion

As we have seen, the truth of van Lambalgen's Theorem and the class of sequences that are low for a certain type of randomness are dependent on the relativization used. It certainly seems that uniform relativization is the proper relativization to consider for all randomness notions. We know that uniform relativization and the standard relativization coincide for Kolmogorov-Loveland randomness and Martin-Löf randomness; for Martin-Löf randomness, at least, the desired results for lowness and van Lambalgen's Theorem will therefore hold regardless of the relativization used. On the other hand, for Schnorr randomness and computable randomness, theorems that ought to be true become true in the context of uniform relativization.

A few words should be said about computable randomness. This is a much more difficult notion to work with than Martin-Löf or Schnorr randomness since its characterization in terms of martingales is the only simple one it has. While Miyabe and Rute have shown that uniformly relative computable randomness makes van Lambalgen's Theorem hold when the standard relativization does not, it is unclear whether a somewhat weaker reducibility would also work. Since we have so few useful characterizations of ways in which a sequence can be far from computable randomness, it is difficult to study relativization in this context. The difficulty of characterizing bases using the standard reducibility suggests, once again, that that is not the correct reducibility to use; however, it does not help us identify a more useful relativization.

There are other components of the study of randomness in which relativization is also important, such as highness: given two classes $\mathcal{C}$ and $\mathcal{D}$ such that $\mathcal{D} \subseteq \mathcal{C}$, a sequence A is high for the pair $(\mathcal{C}, \mathcal{D})$ if $\mathcal{C}^A \subseteq \mathcal{D}$, that is, if A is computationally strong enough to shrink $\mathcal{C}$ enough to fit inside of $\mathcal{D}$. This concept was introduced in [29] and pursued in [2], but only the standard relativizations have ever been considered, and there are still several open questions remaining. We suggest that highness for pairs be more strongly investigated using the standard relativization and that a notion $\mathrm{High}^{\bigstar}(\mathcal{C}, \mathcal{D})$ defined similarly to Kihara and Miyabe's $\mathrm{Low}^{\bigstar}(\mathcal{C}, \mathcal{D})$ be investigated in parallel. Considering the differences in these classes may be very instructive and may provide another avenue to consider the role of relativization in randomness.

References

[1] George Barmpalias and Andrew E. M. Lewis. Chaitin's halting probability and the compression of strings using oracles. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 467(2134):2912–2926, 2011.

[2] George Barmpalias, Joseph S. Miller, and André Nies. Randomness notions and partial relativization. *Israel J. Math.*, 191(2):791–816, 2012.

[3] Bruno Bauwens. Uniform van Lambalgen's theorem fails for computable randomness, 2015.

[4] Benjamín René Callejas Bedregal and André Nies. Lowness properties of reals and hyper-immunity. *Electronic Notes in Theoretical Computer Science*, 84:73–79, 2003.

[5] Laurent Bienvenu, Noam Greenberg, Antonín Kučera, André Nies, and Dan Turetsky. Coherent randomness tests and computing the K-trivial sets. *J. Eur. Math. Soc. (JEMS)*, 18(4):773–812, 2016.

[6] Laurent Bienvenu, Rupert Hölzl, Joseph S. Miller, and André Nies. Denjoy, Demuth and density. *J. Math. Log.*, 14(1):1450004, 35, 2014.

[7] Laurent Bienvenu and Joseph S. Miller. Randomness and lowness notions via open covers. *Ann. Pure Appl. Logic*, 163(5):506–518, 2012.

[8] G. J. Chaitin. Algorithmic information theory. *IBM J. Res. Develop.*, 21(4):350–359, 1977.

[9] S. B. Cooper. Minimal degrees and the jump operator. *J. Symbolic Logic*, 38:249–271, 1973.

[10] Adam R. Day and Joseph S. Miller. Cupping with random sets. *Proc. Amer. Math. Soc.*, 142(8):2871–2879, 2014.

[11] Adam R. Day and Joseph S. Miller. Density, forcing, and the covering problem. *Math. Res. Lett.*, 22(3):719–727, 2015.

[12] R. Downey, D. Hirschfeldt, J. Miller, and A. Nies. Relativizing Chaitin's halting probability. *J. Math. Log.*, 5(2):167–192, 2005.

[13] Rod Downey, Noam Greenberg, Nenad Mihailović, and André Nies. Lowness for computable machines. In *Computational prospects of infinity. Part II. Presented talks*, volume 15 of *Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap.*, pages 79–86. World Sci. Publ., Hackensack, NJ, 2008.

[14] Rod Downey, Evan Griffiths, and Geoffrey LaForte. On Schnorr and computable randomness, martingales, and machines. *Math. Log. Q.*, 50(6):613–627, 2004.
[15] Rod Downey, Denis R. Hirschfeldt, André Nies, and Sebastiaan A. Terwijn. Calibrating randomness. *Bull. Symbolic Logic*, 12(3):411–491, 2006.
[16] Rod Downey, André Nies, Rebecca Weber, and Liang Yu. Lowness and Π^0_2 nullsets. *J. Symbolic Logic*, 71(3):1044–1052, 2006.
[17] Rodney G. Downey and Evan J. Griffiths. Schnorr randomness. *J. Symbolic Logic*, 69(2):533–554, 2004.
[18] Rodney G. Downey, Evan J. Griffiths, and Stephanie Reid. On Kurtz randomness. *Theoret. Comput. Sci.*, 321(2-3):249–270, 2004.
[19] Rodney G. Downey and Denis R. Hirschfeldt. *Algorithmic Randomness and Complexity*. Springer, 2010.
[20] Johanna N.Y. Franklin. Hyperimmune-free degrees and Schnorr triviality. *J. Symbolic Logic*, 73(3):999–1008, 2008.
[21] Johanna N.Y. Franklin. Schnorr trivial reals: A construction. *Arch. Math. Logic*, 46(7–8):665–678, 2008.
[22] Johanna N.Y. Franklin. Lowness and highness properties for randomness notions. In T. Arai et al., editor, *Proceedings of the 10th Asian Logic Conference*, pages 124–151. World Scientific, 2010.
[23] Johanna N.Y. Franklin. Schnorr triviality and genericity. *J. Symbolic Logic*, 75(1):191–207, 2010.
[24] Johanna N.Y. Franklin, Noam Greenberg, Frank Stephan, and Guohua Wu. Anti-complex sets and reducibilities with tiny use. *J. Symbolic Logic*, 78(4):1307–1327, 2013.
[25] Johanna N.Y. Franklin and Keng Meng Ng. Difference randomness. *Proc. Amer. Math. Soc.*, 139(1):345–360, 2011.
[26] Johanna N.Y. Franklin and Reed Solomon. Degrees that are low for isomorphism. *Computability*, 3(2):73–89, 2014.
[27] Johanna N.Y. Franklin and Frank Stephan. Schnorr trivial sets and truth-table reducibility. *J. Symbolic Logic*, 75(2):501–521, 2010.
[28] Johanna N.Y. Franklin and Frank Stephan. Van Lambalgen's Theorem and high degrees. *Notre Dame J. Formal Logic*, 52(2):173–185, 2011.
[29] Johanna N.Y. Franklin, Frank Stephan, and Liang Yu. Relativizations of randomness and genericity notions. *Bull. Lond. Math. Soc.*, 43(4):721–733, 2011.
[30] Johanna N.Y. Franklin and Dan Turetsky. Lowness for isomorphism and degrees of genericity. *Computability*, 7(1):1–6, 2018.
[31] Péter Gács. Every sequence is reducible to a random one. *Inform. and Control*, 70(2-3):186–192, 1986.
[32] Noam Greenberg and Joseph S. Miller. Lowness for Kurtz randomness. *J. Symbolic Logic*, 74(2):665–678, 2009.
[33] Noam Greenberg, Joseph S. Miller, Benoit Monin, and Daniel Turetsky. Two more characterizations of K-triviality. *Notre Dame J. Form. Log.*, 59(2):189–195, 2018.
[34] Denis R. Hirschfeldt, André Nies, and Frank Stephan. Using random sets as oracles. *J. Lond. Math. Soc. (2)*, 75(3):610–622, 2007.
[35] Rupert Hölzl and Wolfgang Merkle. Traceable sets. In *Theoretical computer science*, volume 323 of *IFIP Adv. Inf. Commun. Technol.*, pages 301–315. Springer, Berlin, 2010.
[36] Rupert Hölzl and André Nies. In André Nies, editor, *Logic Blog*.
[37] C. G. Jockusch, Jr., M. Lerman, R. I. Soare, and R. M. Solovay. Recursively enumerable sets modulo iterated jumps and extensions of Arslanov's completeness criterion. *J. Symbolic Logic*, 54(4):1288–1323, 1989.
[38] Takayuki Kihara and Kenshi Miyabe. Uniform Kurtz randomness. *J. Logic Comput.*, 24(4):863–882, 2014.
[39] Takayuki Kihara and Kenshi Miyabe. Unified characterizations of lowness properties via Kolmogorov complexity. *Arch. Math. Logic*, 54(3-4):329–358, 2015.
[40] Bjørn Kjos-Hanssen, Wolfgang Merkle, and Frank Stephan. Kolmogorov complexity and the recursion theorem. *Trans. Amer. Math. Soc.*, 363(10):5465–5480, 2011.
[41] Bjørn Kjos-Hanssen, Joseph S. Miller, and Reed Solomon. Lowness notions, measure and domination. *J. Lond. Math. Soc. (2)*, 85(3):869–888, 2012.
[42] Bjørn Kjos-Hanssen, André Nies, and Frank Stephan. Lowness for the class of Schnorr random reals. *SIAM J. Comput.*, 35(3):647–657, 2005.
[43] Julia F. Knight. Degrees coded in jumps of orderings. *J. Symbolic Logic*, 51(4):1034–1042, 1986.
[44] Antonín Kučera. Measure, Π^0_1-classes and complete extensions of PA. In *Recursion theory week (Oberwolfach, 1984)*, volume 1141 of *Lecture Notes in Math.*, pages 245–259. Springer, Berlin, 1985.

[45] Antonín Kučera. On relative randomness. *Ann. Pure Appl. Logic*, 63(1):61–67, 1993. 9th International Congress of Logic, Methodology and Philosophy of Science (Uppsala, 1991).

[46] Stuart Alan Kurtz. *Randomness and genericity in the degrees of unsolvability.* PhD thesis, University of Illinois, 1981.

[47] Rutger Kuyper and Joseph S. Miller. Nullifying randomness and genericity using symmetric difference. *Ann. Pure Appl. Logic*, 168(9):1692–1699, 2017.

[48] L. A. Levin. Uniform tests for randomness. *Dokl. Akad. Nauk SSSR*, 227(1):33–35, 1976. Russian.

[49] Wolfgang Merkle and Nenad Mihailović. On the construction of effectively random sets. *J. Symbolic Logic*, 69(3):862–878, 2004.

[50] Wolfgang Merkle, Joseph S. Miller, André Nies, Jan Reimann, and Frank Stephan. Kolmogorov-Loveland randomness and stochasticity. *Ann. Pure Appl. Logic*, 138(1-3):183–210, 2006.

[51] Joseph S. Miller. The K-degrees, low for K-degrees, and weakly low for K sets. *Notre Dame J. Form. Log.*, 50(4):381–391 (2010), 2009.

[52] Joseph S. Miller and André Nies. Randomness and computability: Open questions. *Bull. Symbolic Logic*, 12(3):390–410, 2006.

[53] Joseph S. Miller and Liang Yu. On initial segment complexity and degrees of randomness. *Trans. Amer. Math. Soc.*, 360(6):3193–3210, 2008.

[54] Kenshi Miyabe. Truth-table Schnorr randomness and truth-table reducible randomness. *MLQ Math. Log. Q.*, 57(3):323–338, 2011.

[55] Kenshi Miyabe. L^1-computability, layerwise computability and Solovay reducibility. *Computability*, 2(1):15–29, 2013.

[56] Kenshi Miyabe. Schnorr Triviality and Its Equivalent Notions. *Theory Comput. Syst.*, 56(3):465–486, 2015.

[57] Kenshi Miyabe. Reducibilities relating to Schnorr randomness. *Theory Comput. Syst.*, 58(3):441–462, 2016.

[58] Kenshi Miyabe and Jason Rute. van Lambalgen's theorem for uniformly relative Schnorr and computable randomness. In R. Downey et al., editor, *Proceedings of the Twelfth Asian Logic Conference*, pages 251–270. World Scientific, 2013.

[59] André Nies. Lowness properties and randomness. *Adv. Math.*, 197(1):274–305, 2005.

[60] André Nies. Non-cupping and randomness. *Proc. Amer. Math. Soc.*, 135(3):837–844 (electronic), 2007.

[61] André Nies. *Computability and Randomness.* Clarendon Press, Oxford, 2009.

[62] André Nies, Frank Stephan, and Sebastiaan A. Terwijn. Randomness, relativization and Turing degrees. *J. Symbolic Logic*, 70(2):515–535, 2005.

[63] Jan Reimann and Theodore A. Slaman. Measures and their random reals. *Trans. Amer. Math. Soc.*, 367(7):5081–5097, 2015.

[64] Stephen G. Simpson. Almost everywhere domination and superhighness. *MLQ Math. Log. Q.*, 53(4-5):462–482, 2007.

[65] Stephen G. Simpson and Frank Stephan. Cone avoidance and randomness preservation. *Ann. Pure Appl. Logic*, 166(6):713–728, 2015.

[66] Theodore A. Slaman and Robert M. Solovay. When oracles do not help. In *Fourth Annual Workshop on Computational Learning Theory*, pages 379–383. Morgan Kaufman., Los Altos, CA, 1991.

[67] Robert Soare. The Friedberg-Muchnik theorem re-examined. *Canadian J. of Math.*, 24:1070–1078, 1972.

[68] Robert I. Soare. *Turing Computability: Theory and Applications.* Theory and Applications of Computability. Springer, 2016.

[69] Robert M. Solovay. Draft of a paper (or series of papers) on Chaitin's work. Unpublished manuscript, May 1975.

[70] Frank Stephan and Liang Yu. Lowness for weakly 1-generic and Kurtz-random. In *Theory and applications of models of computation*, volume 3959 of *Lecture Notes in Comput. Sci.*, pages 756–764. Springer, Berlin, 2006.

[71] Sebastiaan A. Terwijn and Domenico Zambella. Computational randomness and lowness. *J. Symbolic Logic*, 66(3):1199–1205, 2001.

[72] Michiel van Lambalgen. The axiomatization of randomness. *J. Symbolic Logic*, 55(3):1143–1167, 1990.

[73] Liang Yu. When van Lambalgen's theorem fails. *Proc. Amer. Math. Soc.*, 135(3):861–864, 2007.

[74] Domenico Zambella. On sequences with simple initial segments. Technical Report ILLC ML-1990-05, University of Amsterdam, 1990.

Department of Mathematics, Room 306, Roosevelt Hall, Hofstra University, Hempstead, NY 11549-0114, USA
E-mail address: johanna.n.franklin@hofstra.edu