A SUPERHIGH DIAMOND IN THE C.E. *tt*-DEGREES

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Abstract. The notion of superhigh computably enumerable (c.e.) degrees was first introduced by Mohrherr in [7], where she proved the existence of incomplete superhigh c.e. degrees, and high, but not superhigh, c.e. degrees. Recent research shows that the notion of superhighness is closely related to algorithmic randomness and effective measure theory. Jockusch and Mohrherr proved in [4] that the diamond lattice can be embedded into the c.e. *tt*-degrees preserving 0 and 1 and that the two atoms can be low. In this paper, we prove that the two atoms in such embeddings can also be superhigh.

1. Introduction

In 1966, Lachlan proved that no diamond preserving both 0 and 1 can be embedded in the c.e. Turing degrees [5]. However, Cooper showed that such a diamond can be embedded into the Δ^0_2 degrees if we do not require that the atoms be c.e. [1]. Later, Epstein showed that both atoms can be made to be low or that both atoms can be made to be high [3], and Downey proved in [2] that both atoms can be d.c.e. degrees, giving an extremely sharp result in terms of the Ershov hierarchy.

Alternately, we can consider the possibility of constructing a diamond preserving 0 and 1 if we consider a stronger reducibility. Since the proof of Lachlan's Non-Diamond Theorem holds in the c.e. *wtt*-degrees as well, no such diamond exists in the c.e. *wtt*-degrees. However, Jockusch and Mohrherr showed in [4] that the diamond lattice can be embedded into the c.e. *tt*-degrees preserving 0 and 1 and, furthermore, that the two atoms can be low. In this paper, we present a proof that such a diamond can be embedded into the c.e. *tt*-degrees in such a way that both atoms are superhigh.

The notion of superhigh c.e. degrees was first introduced by Mohrherr in [7], where a computably enumerable set A is defined to be *superhigh* if $A' \equiv_{tt} \emptyset''$. In the same paper, Mohrherr proved the existence of incomplete superhigh c.e. degrees and also the existence of high, but not superhigh, c.e. degrees. More recently, Ng [8] has shown that there is a minimal pair of superhigh c.e. degrees. Recent research in computability theory shows that the notion of superhighness is closely related to algorithmic randomness and effective measure theory. For instance, Simpson showed that uniformly almost everywhere dominating degrees are all superhigh [9]. In fact, it follows that all uniformly almost everywhere dominating degrees are high from Martin's characterization of high sets [6].

Our theorem is stated as follows.

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Theorem 1.1. *There are superhigh computably enumerable sets A and B such that $\mathbf{0}$, $deg_{tt}(A)$, $deg_{tt}(B)$, and $\mathbf{0}'_{tt}$ form a diamond in the computably enumerable tt-degrees.*

Our construction differs from Jockusch and Mohrherr's in several important ways. Jockusch and Mohrherr's construction involves only a finite injury argument, while ours involves an infinite injury argument. This is necessary to make A and B superhigh. Due to this, our sets A and B will not have some of the nice properties that are possessed by the sets constructed by Jockusch and Mohrherr. For instance, they were able to build their atoms A and B with $A \cup B = K$, guaranteeing that $K \equiv_{tt} A \cup B$ in a very obvious way. In our construction, the superhighness strategies will force us to enumerate elements into A and B from time to time to maintain our computations that witness $A' \geq_{tt} TOT$ and $B' \geq_{tt} TOT$. To ensure that $K \leq_{tt} A \oplus B$, we dedicate the numbers of the form $\langle x, 0\rangle$ to meeting this requirement. This allows us to replace Jockusch and Mohrherr's conclusion that $x \in K$ if and only if $x \in A \cup B$ by the slightly more complicated conclusion that $x \in K$ if and only if $\langle x, 0\rangle \in A \cup B$. Again, for the consistency between the superhighness strategies and the minimal pair strategies, we need to be extremely careful when we switch from one outcome to another one.

Our notation and terminology are standard and generally follow Soare [10]. Let φ_e and Φ_e^A be the e-th partial computable function and the e-th A-partial computable function, respectively. In particular, if $\varphi_e(x)\downarrow$, then $[e](x)$ denotes the truth table with index $\varphi_e(x)$ in some effective enumeration of all truth tables, denoted as $\tau_{\varphi_e(x)}$, and $|[e](x)|$ denotes the length of this truth table. For any set A, $[e]^A(x)$ is 0 or 1 depending on whether or not A satisfies the truth table condition with index $\varphi_e(x)$ (denoted by $A \models [e](x)$ if $[e]^A(x) = 1$ and $A \not\models [e](x)$ otherwise). Given two sets A and B, we say that $A \leq_{tt} B$ iff there is an e with φ_e total such that for all x, $[e]^B(x) = A(x)$. When we choose a *fresh* number as a γ-use or a δ-use at stage s, this number is the least number bigger than the corresponding restraint that is not of the form $\langle x, 0\rangle$.

2. Requirements and basic strategies

To prove Theorem 1.1, we will construct two c.e. sets A and B such that both of them are superhigh, K is truth-table reducible to $A \oplus B$, and the tt-degrees of A and B form a minimal pair in the tt-degrees. A and B will satisfy the following requirements:

$\mathcal{P}$: $K \leq_{tt} A \oplus B$;
$\mathcal{S}^A$: $TOT \leq_{tt} A'$;
$\mathcal{S}^B$: $TOT \leq_{tt} B'$;
$\mathcal{N}_{i,j}$: $[i]^A = [j]^B = g$ total $\Rightarrow g$ is computable;

Recall that $TOT = \{e : \varphi_e \text{ is total}\}$ is a Π^0_2-complete set. Therefore, if $\mathcal{S}^A$ and $\mathcal{S}^B$ are satisfied, then A and B will both be superhigh.

2.1. The $\mathcal{P}$-Strategy.

To satisfy the requirement $\mathcal{P}$, we simply code K into $A \oplus B$. We will use a computable enumeration of K such that at each odd stage s, exactly one number, k_s, enters K. At each odd stage s, we will enumerate $\langle k_s, 0\rangle$ into A, B, or both. We will decide which of these sets to enumerate $\langle k_s, 0\rangle$ into based on the actions of the minimal pair strategies $\mathcal{N}_{i,j}$. If $k \notin K$, then $\langle k, 0\rangle$ will never be enumerated into A or B. It is obvious that we will have the equality $K = \{k : \langle k, 0\rangle \in A \cup B\}$, and hence $K \leq_{tt} A \oplus B$.

The $\mathcal{P}$-requirement is global, so we do not put its outcome on the construction tree.

2.2. **An $\mathcal{N}_{i,j}$-Strategy.** Recall that if $[i]$ is a tt-reduction, then for any oracle $X \subseteq \omega$ and any input x, $[i]^X(x)$ converges. The computation $[i]^X(x)$ can be injured at most finitely many times due to the enumeration of numbers less than or equal to $|\tau_{\varphi_i(x)}|$ into X in our construction.

For the requirement $\mathcal{N}_{i,j}$, we apply the diagonalization argument introduced by Jockusch and Mohrherr in [4]. That is, once we see a disagreement between $[i]^A$ and $[j]^B$, we will preserve it forever to make $[i]^A \neq [j]^B$. On the other hand, if $[i]^A$ and $[j]^B$ are equal and total, then we will ensure that they are computable.

Given values for A_s and B_s at stage s, we will define A_{s+1} and B_{s+1} at stage $s+1$ by enumerating more elements into them. Furthermore, if we know that $[i]^A$ and $[j]^B$ differ at k at stage s, we will describe a way of preserving this disagreement at stage $s+1$ even though the enumeration of numbers into A, B, or both might change the computations involved. Let n be a number we want to put into $A_{s+1} \cup B_{s+1}$. There are two cases.

(1) Our number n is of the form $\langle x, 0\rangle$ for some x. Then n is enumerated into A, B, or both for the sake of the requirement $\mathcal{P}$. There are three subcases.

Subcase 1:: If $[i]^{A_s}(k) = [i]^{A_s\cup\{n\}}(k)$, then n is enumerated into A but not into B. Both values are preserved, and the disagreement is preserved as well.

Subcase 2:: If Subcase 1 does not apply but $[j]^{B_s}(k) = [j]^{B_s\cup\{n\}}(k)$, then n is enumerated into B but not into A. As in Case 1, the disagreement is preserved.

Subcase 3:: If $[i]^{A_s}(k) \neq [i]^{A_s\cup\{n\}}(k)$ and $[j]^{B_s}(k) \neq [j]^{B_s\cup\{n\}}(k)$, then n is enumerated into both A and B. In this case, the disagreement is again preserved, as both values are changed.

Note that once one subcase above applies, then we initialize all the strategies with lower priority to avoid the conflict among the $\mathcal{N}$-strategies — obviously, such initializations can happen at most finitely often. We need to be careful here when more $\mathcal{N}$-strategies are considered. It can happen that if we decide to enumerate into A, B, or both, we also need to consider those $\mathcal{N}$-strategies with higher priority, say $\mathcal{N}_{i',j'}$, as we need to avoid the following situation: according to the $\mathcal{N}_{i,j}$-strategy, at stage s_1, a number n_1 is enumerated into A, and at stage s_2, a number n_2 is enumerated into B (corresponding to Subcases 1 and 2, respectively), and such enumerations change $[i']^A(m)$ and $[j']^B(m)$, though separately, and at the next $\mathcal{N}_{i',j'}$-expansionary stage, we may have $[i']^A(m) = [j']^B(m)$, which is different from its original value — $\mathcal{N}_{i',j'}$ is injured.

With this in mind, when we see that the $\mathcal{P}$-strategy wants to enumerate a number into A, B, or both and that an $\mathcal{N}_{i,j}$-strategy is attempting to preserve a disagreement that already exists, instead of automatically implementing the enumeration, we first check whether such an enumeration into A can lead to a disagreement between $[i']^A$ and $[j']^B$. If not, then we just work as described above (in Subcase 3, we now enumerate n into B and check whether this enumeration into B can lead to a disagreement for $\mathcal{N}_{i',j'}$ — here n is enumerated into A and B separately). Otherwise, we start to preserve this disagreement to satisfy $\mathcal{N}_{i',j'}$ — the $\mathcal{N}_{i,j}$ considered above is initialized, and again, even if Subcase 3 applies, we do not enumerate n into B.

(2) Our number n is a number chosen by an $\mathcal{S}_e^A$-strategy or an $\mathcal{S}_e^B$-strategy. Without loss of generality, suppose that n is selected by an $\mathcal{S}_e^A$-strategy and we want to put it into A. As in the standard construction of high sets, we will consider "believable" computations to allow us to handle the potential infinitary injury to the negative strategies caused by the higher priority highness strategies; for instance, $[i]^A(m)$. In this way, when we see $[i]^A$ and $[j]^B$, if this $\mathcal{S}_e^A$-strategy has higher priority than $\mathcal{N}_{i,j}$, then the enumeration of n into A does not affect the computation $[i]^A(m)$. This will be described further in the discussion of the $\mathcal{S}_e^A$-strategies below. The notion of a believable computation will be defined formally in Definition 2.1.

An $\mathcal{N}_{i,j}$-strategy has three outcomes: ∞, f and d, where ∞ denotes that there are infinitely many expansionary stages, f denotes that there are only finitely many expansionary stages, but no disagreement is produced, and d denotes that a disagreement between $[i]^A$ and $[j]^B$ is produced and preserved successfully.

2.3. **An $\mathcal{S}_e^A$-Strategy.** To make A superhigh, instead of giving a truth-table reduction from TOT to A' explicitly, we will construct a binary functional $\Gamma^A(e,x)$ such that for all $e \in \omega$,

$$\mathrm{TOT}(e) = \lim_{x\to\infty} \Gamma^A(e,x)$$

with $|\{x : \Gamma^A(e,x) \neq \Gamma^A(e,x+1)\}|$ bounded by a computable function h, which will ensure that $\mathrm{TOT} \leq_{tt} A'$. The relativized Limit Lemma will guarantee that A will be superhigh. (In the case of B, we will construct a binary functional $\Delta^B(e,y)$ satisfying a similar requirement.) The crucial point is to find this computable bounding function h.

As usual, $\mathcal{S}^A$ is divided into infinitely many substrategies $\mathcal{S}_e^A$, $e \in \omega$, each of which has two outcomes, ∞ (a Π_2^0-outcome) and f (a Σ_2^0-outcome), where ∞ denotes the guess that φ_e is total and f denotes the guess that φ_e is not total.

Let β be an $\mathcal{S}_e^A$-strategy on the priority tree. As usual, we have the following standard definition of length agreement function:

$$l(\beta,s) = \max\{\ x < s : s \text{ is a } \beta\text{-stage and } \varphi_e(y)[s]\downarrow \ \text{ for all } \ y < x\};$$

$$m(\beta,s) = \max\{\ l(\beta,t) : t < s \text{ is an } \beta\text{-stage}\};$$

where t is a β-stage if β is visited at stage t. Say that s is a β-expansionary stage if $s = 0$ or $l(\beta,s) > m(\beta,s)$.

Let s be a β-stage. If s is a β-expansionary stage, then we believe that φ_e is total, and we undefine every $\Gamma^A(e,x)$ defined by lower priority strategies by enumerating the corresponding $\gamma(e,x)$ into A and then define $\Gamma^A(e,y)$ to be 1 for the least y such that $\Gamma^A(e,y)$ is undefined. If s is not a β-expansionary stage, then we believe that φ_e is not total, and again, we undefine those $\Gamma^A(e,x)$ defined by lower priority strategies by enumerating the corresponding $\gamma(e,x)$ into A and then define $\Gamma^A(e,y)$ to be 0 for the least y with $\Gamma^A(e,y)$ not defined. Thus, if there are infinitely many β-expansionary stages (so φ_e is total, $e \in \mathrm{TOT}$, and ∞ is the true outcome of β), then $\Gamma^A(e,x)$ is defined as 1 for almost all $x \in \omega$. On the other hand, if there are only finitely many β-expansionary stages (so φ_e is not total, $e \notin \mathrm{TOT}$, and f is the true outcome of β), then $\Gamma^A(e,x)$ is defined as 0 for almost all $x \in \omega$.

Thus, for a fixed $\mathcal{S}_e^A$-strategy β on the construction tree, β will attempt to redefine $\Gamma^A(e,x)$ for almost all x, with the exception that (a) some γ-uses are prevented from being enumerated into A by higher priority strategies (when a disagreement is produced), or (b) $\Gamma^A(e,x)$ is defined by another $\mathcal{S}_e^A$-strategy with higher priority. In particular, if β is the $\mathcal{S}_e^A$-strategy on the true path, then there are only finitely many strategies with higher priority that can be visited during the whole construction, and hence β will succeed in defining $\Gamma^A(e,x)$ for almost all x.

Suppose that β is on the true path and n is the length of β. We will see that $|\{x : \Gamma^A(e,x) \neq \Gamma^A(e,x+1)\}| \leq 2^{3^{n+1}}$. To see this, note that (a) above can happen at most 2^{3^n} times, as there are at most 3^n many strategies with length less than n, and each time when one of them produces (not preserves) a disagreement, a restraint is set, preventing α from rectifying $\Gamma^A(e,x)$ for some x. Note that after an $\mathcal{N}$-strategy α produces a disagreement, say at stage s, whenever α requires us to preserve this disagreement, all the strategies with lower priority will be initialized, and at the same time, all of the γ-uses and δ-uses defined after stage s will be enumerated into A and B respectively (one by one, as pointed out above, for the sake of the $\mathcal{N}$-strategies with priority higher than α). It is crucial for us to ensure that TOT is truth-table reducible to A' and B', as we will discuss below. Here, when β is initialized by a strategy with higher priority with length $\geq n$, an $\mathcal{S}_e^A$-strategy β' on the left of β is visited, and β' takes the responsibility of rectifying $\Gamma^A(e,x)$ for some x, which can lead to an inequality between $\Gamma^A(e,x)$ and $\Gamma^A(e,x+1)$. Thus, (b) can happen at most 3^n many times. In total, the number of those x such that β cannot rectify $\Gamma^A(e,x)$ is at most $2^{3^{n+1}}$, which ensures that $Tot \leq_{tt} A'$, where the corresponding bounding function h is given by $h(e) = 2^{3^{e+1}}$.

We have seen some interactions between the $\mathcal{P}$-strategy and the $\mathcal{N}$-strategies. Now we describe the interactions between the $\mathcal{N}$-strategies, the $\mathcal{S}$-strategies, and the $\mathcal{P}$-strategy.

Assume that α is an $\mathcal{N}_{i,j}$-strategy, β is an $\mathcal{S}_e^A$-strategy, and ζ is an $\mathcal{S}_{e'}^B$-strategy with $\beta^\frown\infty \subseteq \zeta^\frown\infty \subseteq \alpha$. The following may happen: at a stage s, a disagreement between $[i]^A$ and $[j]^B$ appears at α, so α wants to preserve this disagreement by initializing all of strategies with lower priority. However, this disagreement can be destroyed by β and ζ, as they may enumerate small γ-uses and δ-uses into A and B separately. To avoid this, we require only that α recognizes α-believable computations, defined formally below.

Definition 2.1. Let α be an $\mathcal{N}_{i,j}$-strategy, and β be an $\mathcal{S}_e^A$-strategy with $\beta^\frown\infty \subseteq \alpha$.

(1) A computation $[i]^{A_s}(m)$ is α-believable at β at stage s if for each x with $\gamma(e,x)[s]$ defined by β and less than the length of the truth-table of $[i](m)$, $\Gamma^{A_s}(e,x)[s]$ is equal to 1.

(2) A computation $[i]^{A_s}(m)$ is α-believable at stage s if it is α-believable at β at stage s for any $\mathcal{S}_e^A$-strategy β, $e \in \omega$, with $\beta^\frown\infty \subseteq \alpha$.

We can define an α-believable computation $[j]^{B_s}(m)$ similarly.

We are ready to define an α-expansionary stage for an $\mathcal{N}_{i,j}$-strategy α.

Definition 2.2. Let α be an $\mathcal{N}_{i,j}$-strategy. The length of agreement between $[i]^A$ and $[j]^B$ is defined as follows:

$$l(\alpha,s) = \max\{x < s : \text{for all } y < x,\ [i]^A(y)[s] = [j]^B(y)[s]$$

via α-believable computations}.
$m(\alpha, s) = \max\{l(\alpha, t) : t < s \text{ is an } \alpha\text{-stage}\}$.
Say that a stage s is α-expansionary if $s = 0$ or $l(\alpha, s) > m(\alpha, s)$.

Now we consider the situation when β, an $\mathcal{S}_e^A$-strategy, changes its outcome from f to ∞ at a β-expansionary stage. Let s' be the last β-expansionary stage. Unlike the construction of high degrees, to make A and B superhigh, we need to enumerate all the γ-uses and δ-uses defined by strategies below outcome f between stages s' and s into A and B. This enumeration also takes place when the outcome of an $N_{i,j}$ node moves from f to ∞. Again, these numbers cannot be enumerated into A and B simultaneously, as discussed above in the section on $\mathcal{N}$-strategies, for the sake of $\mathcal{N}$-strategies with priority higher than β. Let F_β^A and F_β^B be the collections of these γ-uses and δ-uses respectively. We put the numbers in $F_\beta^A \cup F_\beta^B$ into A or B correspondingly, one by one, from the least to the greatest, and whenever one number is enumerated, we reconsider the $\mathcal{N}$-strategies with higher priority to see whether a disagreement appears. Once such a disagreement appears at an $\mathcal{N}$-strategy, say α, we stop the enumeration as we need to satisfy α via this disagreement. In this case, β is injured. Note that β can be injured in this way only by those $\mathcal{N}$-strategies α such that $\alpha \subset \beta$. We will refer to this enumeration process as an "*outcome-shifting enumeration process*" for simplicity.

2.4. **Construction.** First, we define the priority tree T and assign requirements to the nodes on T as follows. Suppose $\sigma \in T$. If $|\sigma| = 3e$, then σ is assigned to the $\mathcal{N}_{i,j}$-strategy such that $e = \langle i, j\rangle$. It has three possible outcomes: ∞, f, and d, with $\infty <_L f <_L d$. If $|\sigma| = 3e+1$, then σ is assigned to the $\mathcal{S}_e^A$-strategy. If $|\sigma| = 3e+2$, then σ is assigned to the $\mathcal{S}_e^B$-strategy. In the latter two cases, σ has two possible outcomes: ∞ and f, with $\infty <_L f$.

$\mathcal{P}$ is a global requirement, and we do not put it on the tree.

We assume that K is enumerated at odd stages. That is, we fix an enumeration $\{k_{2s+1}\}_{s\in\omega}$ of K such that at each odd stage $2s+1$, exactly one number, k_{2s+1}, is enumerated into K.

In the construction, we say that an $\mathcal{N}_{i,j}$-strategy α *sees a disagreement at k at a stage s* if $k \le s$, $[i]^{A_s}$ and $[j]^{B_s}$ agree on all arguments $\le k$, and one of the following cases applies:

(i) s is odd (k_s enters K and we need to put $\langle k_s, 0\rangle$ into $A \cup B$). In this case, either
   (1) $[i]^{A_s}(k) \neq [i]^{A_s \cup \{\langle k_s, 0\rangle\}}(k)$,
   (2) $[j]^{B_s}(k) \neq [j]^{B_s \cup \{\langle k_s, 0\rangle\}}(k)$, or
   (3) a disagreement is produced by the enumeration of the γ- or δ-uses into A or B by the initialization. For instance, there may be an $\mathcal{N}$-strategy $\alpha' \supset \alpha$ that attempts to preserve a disagreement, and the enumeration of $\langle k_s, 0\rangle$ into A, B, or both (depending on α') and a one-by-one enumeration of elements of $F_A \cup F_B$ into A and B (in increasing order, as described in the $\mathcal{S}$-strategies) would lead to either $[i]^A(k) \neq [i]^{A_s}(k)$ or $[j]^B(k) \neq [j]^{B_s}(k)$. Here, F_A and F_B are the finite collections of γ-uses and δ-uses defined below outcome $\alpha'^\frown d$ after the last stage α' that produces or preserves its disagreement.

If (1) is true, then we enumerate $\langle k_s, 0\rangle$ into A. If (1) is not true but (2) is, then we enumerate $\langle k_s, 0\rangle$ into B. Otherwise, (3) is true, and we enumerate $\langle k_s, 0\rangle$ into A or B or both, according to α'. We also enumerate the corresponding numbers in $F_A \cup F_B$ into A and B respectively.

As a consequence, a disagreement between $[i]^A(k)$ and $[j]^B(k)$ is produced, and α will preserve this disagreement forever unless it is initialized later.

(ii) s is even (s is a β-expansionary stage for some $\mathcal{S}$-strategy β).

Let β be such a strategy, and let s' be the last β-expansionary stage. At stage s, to change its outcome from f to ∞, we need to enumerate all of the elements in F_A and F_B into A and B respectively. Here, F_A and F_B are the finite collections of γ-uses and δ-uses defined below outcome $\beta^\frown f$ after stage s'. Again, we enumerate these numbers into A and B in increasing order until we find that either $[i]^A(k) \neq [i]^{A_s}(k)$ or $[j]^B(k) \neq [j]^{B_s}(k)$ is true; that is, until a disagreement between $[i]^A(k)$ and $[j]^B(k)$ is produced. From now on, α will preserve this disagreement forever unless it is initialized later.

We recall that an $\mathcal{N}_{i,j}$-strategy α *preserves a disagreement at k at an odd stage s* if this disagreement was produced before and has been preserved so far (so $[i]^{A_s}(k) \neq [j]^{B_s}(k)$) and $\langle k_s, 0\rangle$ is less than one of the lengths of the truth-tables $[i](k)$ and $[j](k)$. Enumerating $\langle k_s, 0\rangle$ into $A \cup B$ causes one of the following to happen:

1. If $[i]^{A_s}(k) = [i]^{A_s \cup \{\langle k_s,0\rangle\}}(k)$, then $\langle k_s, 0\rangle$ is enumerated into A but not into B. Both values are preserved, and the disagreement is preserved as well.
2. If $[j]^{B_s}(k) = [j]^{B_s \cup \{\langle k_s,0\rangle\}}(k)$, then $\langle k_s, 0\rangle$ is enumerated into B but not into A. As in Case 1, the disagreement is preserved.
3. If $[i]^{A_s}(k) \neq [i]^{A_s \cup \{\langle k_s,0\rangle\}}(k)$ and $[i]^{B_s}(k) \neq [i]^{B_s \cup \{\langle k_s,0\rangle\}}(k)$, then $\langle k_s, 0\rangle$ is enumerated into both A and B. In this case, the disagreement is again preserved, as both values are changed.

Note that whenever α produces or preserves a disagreement in this manner, all the strategies below the outcome $\alpha^\frown d$ are initialized. Such initializations can happen at most finitely often.

Formal Description of the Construction.

Stage 0: Initialize all the nodes on T and set $A_0 = B_0 = \emptyset$. Let $\Gamma^A(e,x)[0]$ and $\Delta^B(e,x)[0]$ be undefined for each e and x.

Stage $s > 0$:

Case 1: *s is odd.* We will put $\langle k_s, 0\rangle$ into $A \cup B$ at this stage.

First check whether there is an $\mathcal{N}$-strategy that sees a disagreement or needs to preserve a disagreement. Let α be the highest priority such $\mathcal{N}$-strategy. Enumerate $\langle k_s, 0\rangle$ into A or B or both accordingly. Initialize all the strategies with lower priority and do the corresponding enumerations as in (i). Otherwise, we just enumerate $\langle k_s, 0\rangle$ into A and go to the next stage.

Case 2: *s is even.* We define the approximation to the true path σ_s of length $\leq s$. Suppose that $\sigma_s \upharpoonright u$ has been defined for $u < t$ and let ξ be $\sigma_s \upharpoonright t$. We will define $\sigma_s(t)$. We have the following two subcases.

Subcase 1: ξ is an $\mathcal{N}_{i,j}$-strategy for some i and j. If ξ has produced a disagreement before and ξ has not been initialized since then, we let $\sigma_s(t) = d$. Otherwise, we check whether s is a ξ-expansionary stage. If not, then let $\sigma_s(t) = f$. If it is, then we start the outcome-shifting enumeration process to enumerate those γ-uses from F_A and δ-uses from F_B defined below the outcome $\xi^\frown f$ from the last ξ-expansionary stage into A and B respectively, one by one and in increasing order. At the same time, each time we enumerate such a number, we check whether there is an $\mathcal{N}$-strategy $\alpha \subset \xi$ that can produce a disagreement. If there is, then we stop the enumeration of F_A and F_B into A and B and let $\sigma_s = \alpha$. Declare that α produces a disagreement at stage s, let $\sigma_s = \alpha$, and go to the 'defining' phase. If not, then after all numbers in $F_A \cup F_B$ have been enumerated, we let $\sigma_s(t) = \infty$ and go to the next substage.

Subcase 2: ξ is an $\mathcal{S}^A_e$-strategy or an $\mathcal{S}^B_e$-strategy for some e. If s is not a ξ-expansionary stage, let $\sigma_s(t) = f$ and go to the next substage. Otherwise, we start the outcome-shifting enumeration process as described in Subcase 1.

Defining Phase of stage s: For those $\mathcal{S}^A_e$-strategies β with $\beta^\frown\infty \subseteq \sigma_s$, find the least y such that $\Gamma^A(e, y)$ is currently not defined, define it as 1 and let the use $\gamma(e, y)$ be a fresh number, and for those $\mathcal{S}^A_e$-strategies β with $\beta^\frown f$, find the least y such that $\Gamma^A(e, y)$ is currently not defined, define it as 0, and let the use $\gamma(e, y)$ be a fresh number. For those $\mathcal{S}^B_e$-strategies β, we define $\Delta^B(e, y)$ in the same way. Initialize all the strategies with lower priority than σ_s and go to the next stage.

Note that the enumeration of those γ-uses and δ-uses at substages into A and B ensures that those $\Gamma^A(e, x)$ and $\Delta^B(e, y)$ defined by those strategies with priority lower than σ_s are undefined.

This completes the construction.

2.5. **Verification.** Let $TP = \liminf_s \sigma_{2s}$ be the true path of the construction. We first prove that TP is infinite and then verify that the construction given above satisfies all the requirements. First, by the actions at the odd stages, we have that

$$k \in K \iff \langle k, 0\rangle \in A \cup B,$$

and hence

Lemma 2.3. $K \leq_{\mathrm{tt}} A \oplus B$.

The following lemma says that TP is infinite.

Lemma 2.4. *Let ς be any node on TP. Then*

(1) *ς can only be initialized finitely often.*
(2) *ς can initialize strategies with lower priority at most finitely often.*
(3) *ς has an outcome $\mathcal{O}$ such that $\varsigma^\frown\mathcal{O}$ is on TP.*

Proof. We prove this lemma by induction. Let ς^- be the immediate predecessor of ς. By the induction hypothesis, there is a least stage s_0 after which ς^- can never be initialized again. Also assume that $\varsigma = \varsigma^{-\frown}\mathcal{O}'$. There are two cases.

Case 1: $\varsigma^- = \alpha$ is an $\mathcal{N}_{i,j}$-strategy for some $i, j \in \omega$.

If $\mathcal{O}'$ is d, then after stage s_0, α produces a disagreement, and this disagreement can never be destroyed as all the strategies with lower priority are initialized when this disagreement is produced. Therefore, after this, α initializes ς only when it preserves this disagreement, which can happen at most finitely often. This means that after a stage large enough that A and B have been fixed on the numbers involved in the truth-tables involved, whenever ς^- is visited at a stage s, ς is also visited at this stage, and (1) is true for ς.

If $\mathcal{O}'$ is f, then after stage s_0, there are at most finitely many α-expansionary stages. Let $s \geq s_0$ be the last α-expansionary stage. Then, after stage s, whenever ς^- is visited at a stage s, ς is also visited at this stage, and hence $\alpha^\frown\infty$ will not be visited again and ς cannot be initialized by $\alpha^\frown\infty$. If this happens, (1) is true. Note that in this case, after stage s_0, α cannot produce any disagreements as otherwise it would be preserved forever and α would have outcome d.

If $\mathcal{O}'$ is ∞, then by the choice of stage s_0, α cannot produce any disagreement after stage s_0, and ς cannot be initialized. Again, (1) is true.

(2) is obviously true since ς is an $\mathcal{S}$-strategy, which do not initialize lower priority strategies at all.

Since ς has only three outcomes, let $\mathcal{O}$ be the leftmost one that is visited infinitely often. By the construction, we never terminate the definition of σ_s at ς itself (since ς is an $\mathcal{S}$-node). Hence at almost every ς-stage (even stage), we will be able to continue the definition of σ_s beyond ς. Therefore, (3) is also true for ς.

Case 2: $\varsigma^- = \beta$ is an $\mathcal{S}^A_e$-strategy for some $e \in \omega$.

If $\mathcal{O}'$ is f, then after stage s_0, there are at most finitely many α-expansionary stages. Let $s \geq s_0$ be the last α-expansionary stage. Then after stage s, whenever β is visited, ς is also visited at this stage, and hence $\beta^\frown\infty$ cannot be visited again and ς cannot be initialized by $\beta^\frown\infty$. If $\mathcal{O}'$ is ∞, then by the choice of stage s_0, ς cannot be initialized after stage s_0. Therefore, (1) is true in both cases.

(2) is true as ς is also an $\mathcal{S}$-strategy.

(3) is true for ς for the same reason as in Case 1.

Case 3: $\varsigma^- = \beta$ is an $\mathcal{S}^B_e$-strategy for some e.

(1) is true by the same argument given in Case 2. To see (2), observe that after stage s_0, if ς does not produce any disagreement, then it does not initialize strategies with lower priority at all. Otherwise, if ς produces a disagreement at a stage $s \geq s_0$, then after stage s_0, ς can initialize at most finitely often to preserve this disagreement, and it will not initialize after some sufficiently large stage.

If (3) holds, then ς is an $\mathcal{N}_{i,j}$-strategy. By the construction, after stage s_0, any disagreement produced is preserved forever. Producing disagreement can happen at most one time, so the construction σ_s stops at ς by producing disagreement at most once. After a disagreement is produced, only preserving the disagreement can stop the construction σ_s at ς. However, there are at most finitely times that the existing disagreement must be preserved, so we will terminate the construction σ_s at ς only finitely often. Therefore, (3) is also true. □

From Lemma 2.4, we can see that any $\mathcal{N}$-strategy on TP is satisfied, and hence

Lemma 2.5. *For all $i, j \in \omega$, the requirement $\mathcal{N}_{i,j}$ is satisfied.*

Proof. Fix i and j, and let σ be the $\mathcal{N}_{i,j}$-strategy on TP. Also suppose that $[i]^A = [j]^B$ is total. We prove that $[i]^A$ is computable. Let s_0 be the last stage at which σ is initialized.

First, note that after stage s_0, σ does not produce any disagreement at all, as otherwise, as described in Lemma 2.4, the disagreement will be preserved forever, and hence $[i]^A \neq [j]^B$.

Fix n. We compute $[i]^A(n)$ by looking at the first σ-expansionary stage $s > s_0$ with $l(\sigma, s) > n$. We claim that $[i]^A(n) = [i]^{A_s}(n)$ and that $[j]^B(n) = [j]^{B_s}(n)$. Suppose not, and assume that $[i]^{A_s}(n) \neq [i]^{A_{s'}}(n)$ for some least $s' > s$ (without loss of generality, we assume that the A-side changes first). Then at stage s', we will see such a possible disagreement, and σ will produce and preserve this disagreement forever. This will contradict our assumption. □

The next lemma shows that all $\mathcal{S}$-strategies are satisfied.

Lemma 2.6. *For any $e \in \omega$, $\mathcal{S}_e^A$ and $\mathcal{S}_e^B$ are satisfied.*

Proof. Fix e. We prove that $\mathcal{S}_e^A$ and $\mathcal{S}_e^B$ are satisfied in exactly the same way. First, we show that $\mathrm{TOT}(e) = \lim_{x\to\infty} \Gamma^A(e, x)$.

Let β be the $\mathcal{S}_e^A$-strategy on TP, and let s_0 be the last stage at which β is initialized. By our construction, the $\Gamma^A(e, x)$ which are defined by β after stage s_0 are never undefined by another strategy. In fact, β defines $\Gamma^A(e, x)$ for almost all x. That is, after stage s_0, whenever β is visited, those $\Gamma^A(e, x)$ defined by strategies on the right of β are undefined, as these γ-uses have been enumerated into A. This ensures that Γ^A is total. If there are only finitely many β-expansionary stages, then after a sufficiently large stage, β defines $\Gamma^A(e, x)$ only as 0, which ensures that $\lim_x \Gamma^A(e, x) = 0$ and hence is equal to $\mathrm{TOT}(e)$. If there are infinitely many β-expansionary stages, then after stage s_0, at each β-expansionary stage, β succeeds in enumerating all those $\gamma(e, -)$-uses defined under the outcome f into A and redefines $\Gamma^A(e, x)$ to be 1, which ensures that $\lim_x \Gamma^A(e, x) = 1$ and hence is equal to $\mathrm{TOT}(e)$.

Now we show that $|\{x : \Gamma^A(e, x) \neq \Gamma^A(e, x + 1)\}| \leq 2^{3^{e+1}}$. Note that in the construction, $\Gamma^A(e, x)$ may not be equal to $\Gamma^A(e, x + 1)$ for some x, since a ς-strategy defining $\Gamma^A(e, x)$ is initialized when an $\mathcal{N}_{i,j}$-strategy with higher priority produces a disagreement or $\Gamma^A(e, x)$ is defined by an $\mathcal{S}_e^A$-strategy on the left of β. There are at most 3^e many such $\mathcal{N}$-strategies, and once a disagreement is produced by such an $\mathcal{N}$-strategy, it is preserved unless it is initialized later. By a simple counting argument, we know that there are at most 2^{3^e} such initializations, and hence there are no more than $2^{3^{e+1}}$ many x such that $\Gamma^A(e, x) \neq \Gamma^A(e, x + 1)$. □

By Lemma 2.6, we have

Lemma 2.7. $\mathrm{TOT} \leq_{tt} A', B'$ *and hence A and B are superhigh.*

This completes the proof of Theorem 1.1.

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